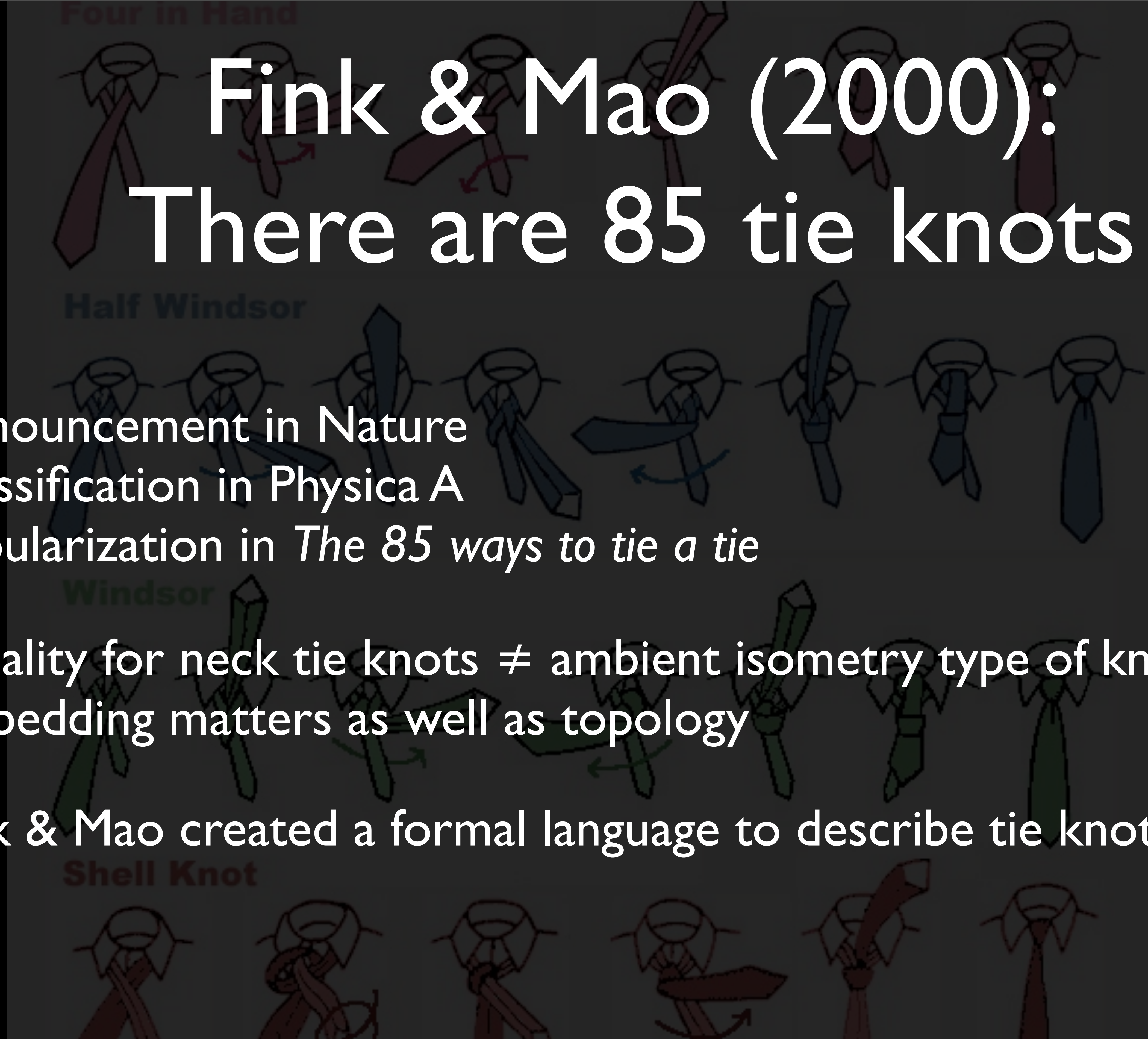
On tie knots

Mikael Vejdemo-Johansson
Anders Sandberg, Meredith L. Patterson, Dan Hirsh



Fink & Mao (2000): There are 85 tie knots

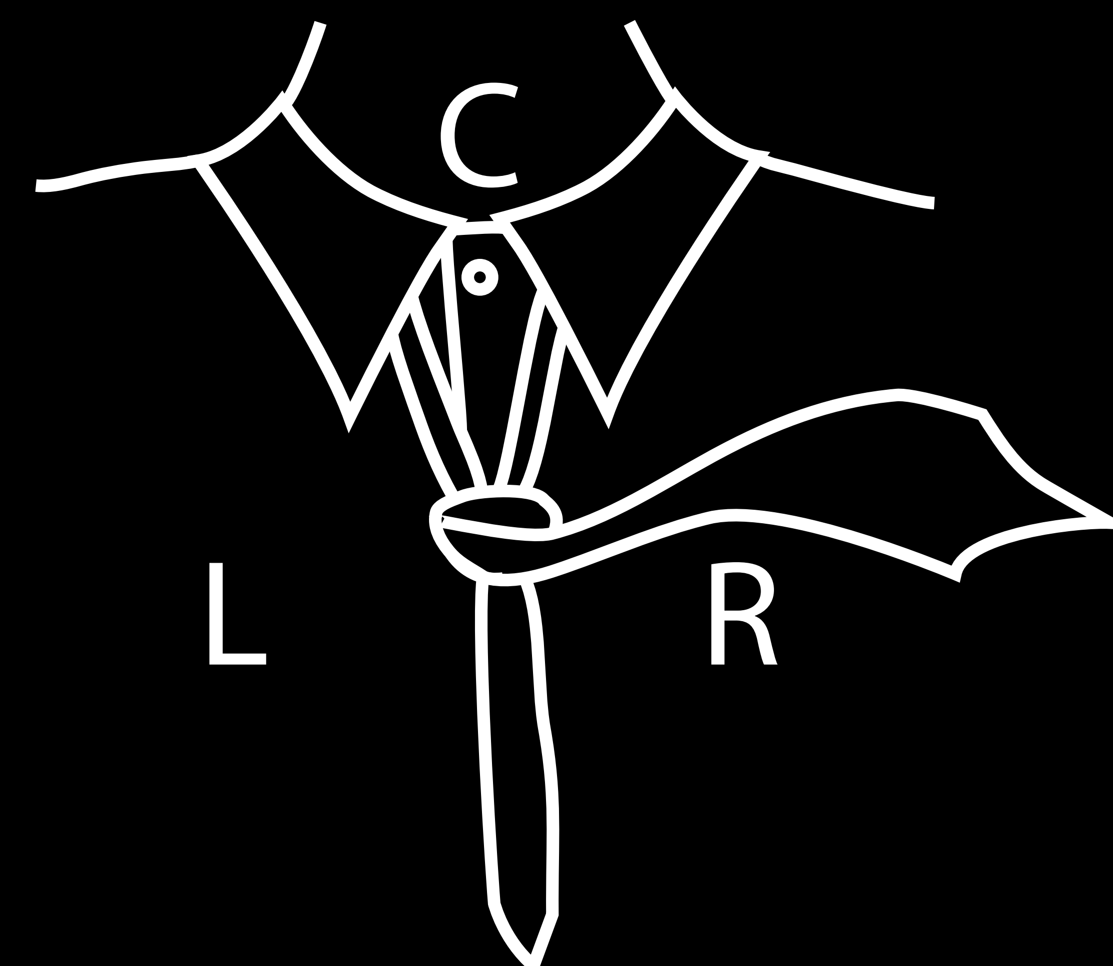
- Announcement in Nature
  Classification in Physica A
  Popularization in *The 85 ways to tie a tie*

- Equality for neck tie knots ≠ ambient isometry type of knot: Embedding matters as well as topology

- Fink & Mao created a formal language to describe tie knot sequences

A formal language for tie knots
Fink & Mao

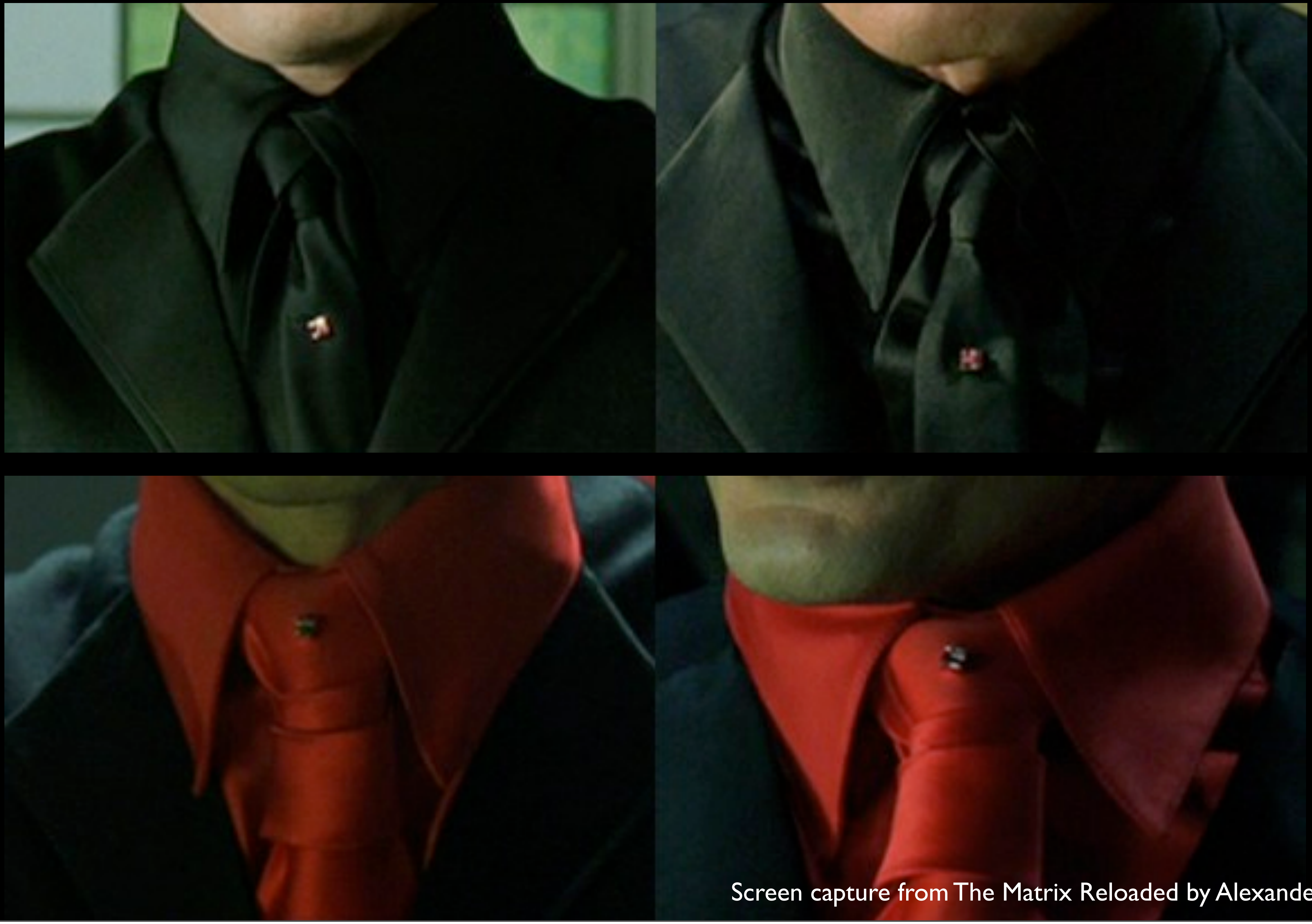
- Symbols denoting region and direction; and a special symbol U for tucking the tie under itself.
- Each region symbol adorned with ⊙ or ⊗ to denote direction.
- Example:
  4-in-hand is $L_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}U$
  Trinity is $L_{\otimes}C_{\odot}L_{\otimes}R_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}UR_{\otimes}L_{\odot}U$



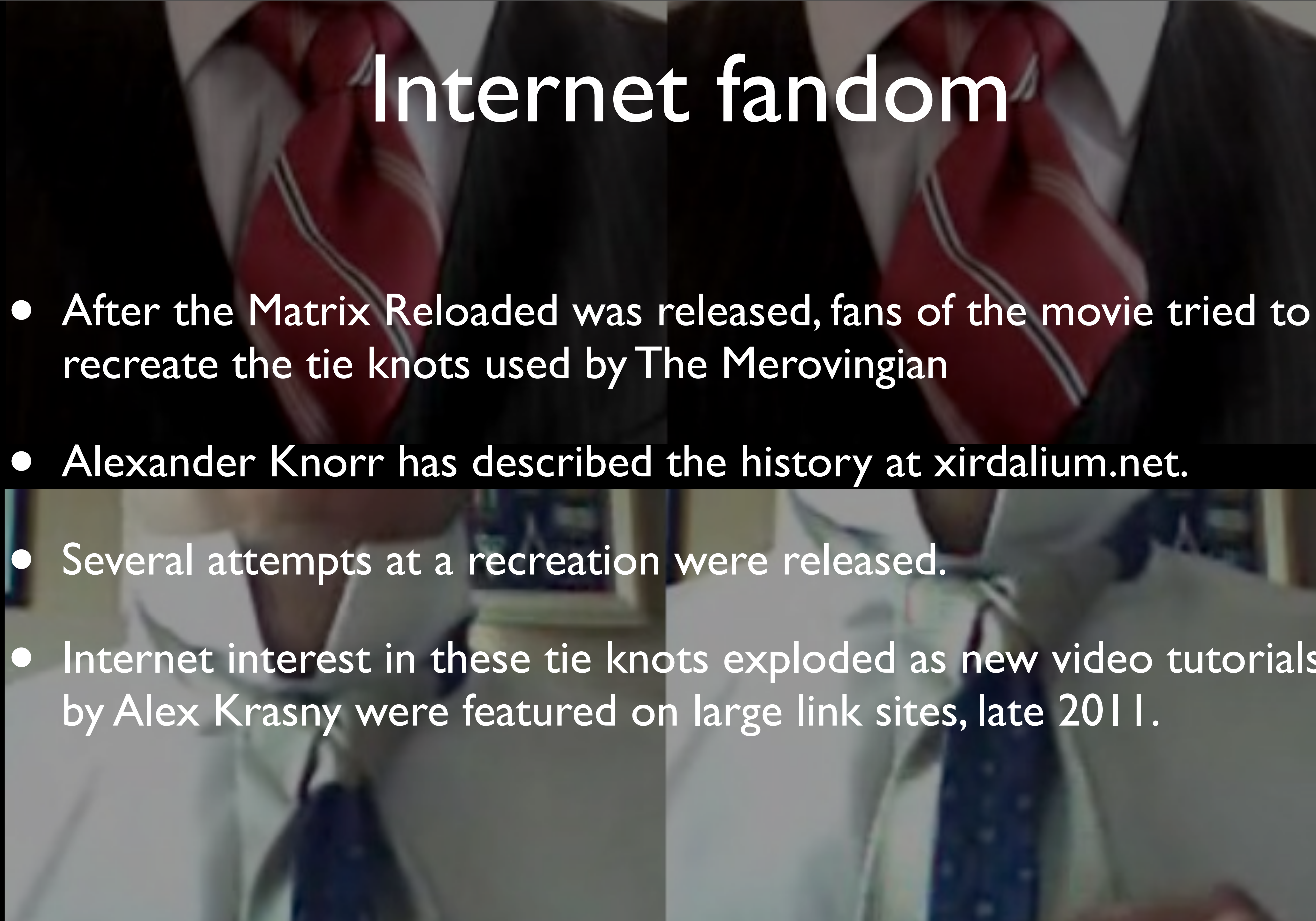
Axioms for Fink & Mao's language

- No two subsequent symbols share the same {L,R,C}
- No two subsequent symbols share the same {⊗, ⊙}
- All tie knots end with either $L_{\odot}R_{\otimes}C_{\odot}U$ or $R_{\odot}L_{\otimes}C_{\odot}U$
- U is only valid after a ⊙-move

Matrix Reloaded — a non-tie-knot



Screen capture from The Matrix Reloaded by Alexander Knorr (xirdalium)



Internet fandom

- After the Matrix Reloaded was released, fans of the movie tried to recreate the tie knots used by The Merovingian
- Alexander Knorr has described the history at xirdalium.net.
- Several attempts at a recreation were released.
- Internet interest in these tie knots exploded as new video tutorials by Alex Krasny were featured on large link sites, late 2011.

These knots are not among the 85

By design: none of these knots have flat frontal *façades*.

Thin blade knots: modifying Fink-Mao's language

- Comparing the novel knot families with Fink & Mao's formal language, we have:
  - ✳ weakened their assumptions, including these new knots
  - ✳ simplified their language
  - ✳ extended their enumeration
  - ✳ classified their language in the Chomsky hierarchy of computational complexity for formal languages

Thin blade knots: modifying Fink-Mao's language

- *No two subsequent symbols share the same {⊗, ⊙}*
  U is only valid after a ⊙-move

  So ⊗, ⊙ are irrelevant; we can deduce these from the length of a knot description.

- *No two subsequent symbols share the same {L,R,C}*

  So only the direction (L → R → C or L → C → R) matters

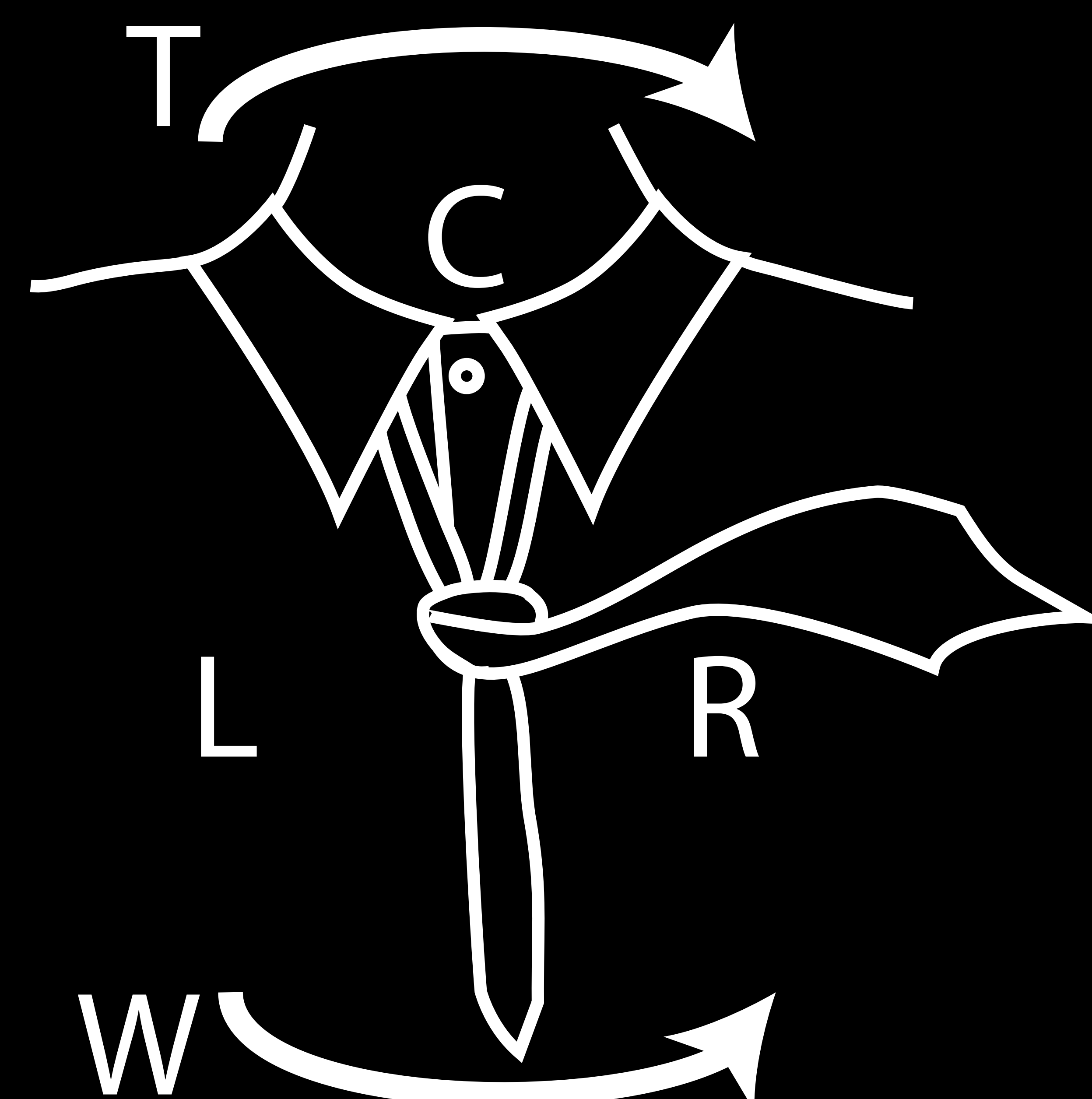
- ~~*All tie knots end with either $L_{\odot}R_{\otimes}C_{\odot}U$ or $R_{\odot}L_{\otimes}C_{\odot}U$*~~

Introducing new symbols

- We introduce a new alphabet: {T, W, U}

  For **T**urnwise, **W**iddershins and **U**nde

- Arbitrary strings of T/W form valid winding sequences
- Tucking the tie under depends on the existence of something to tuck under

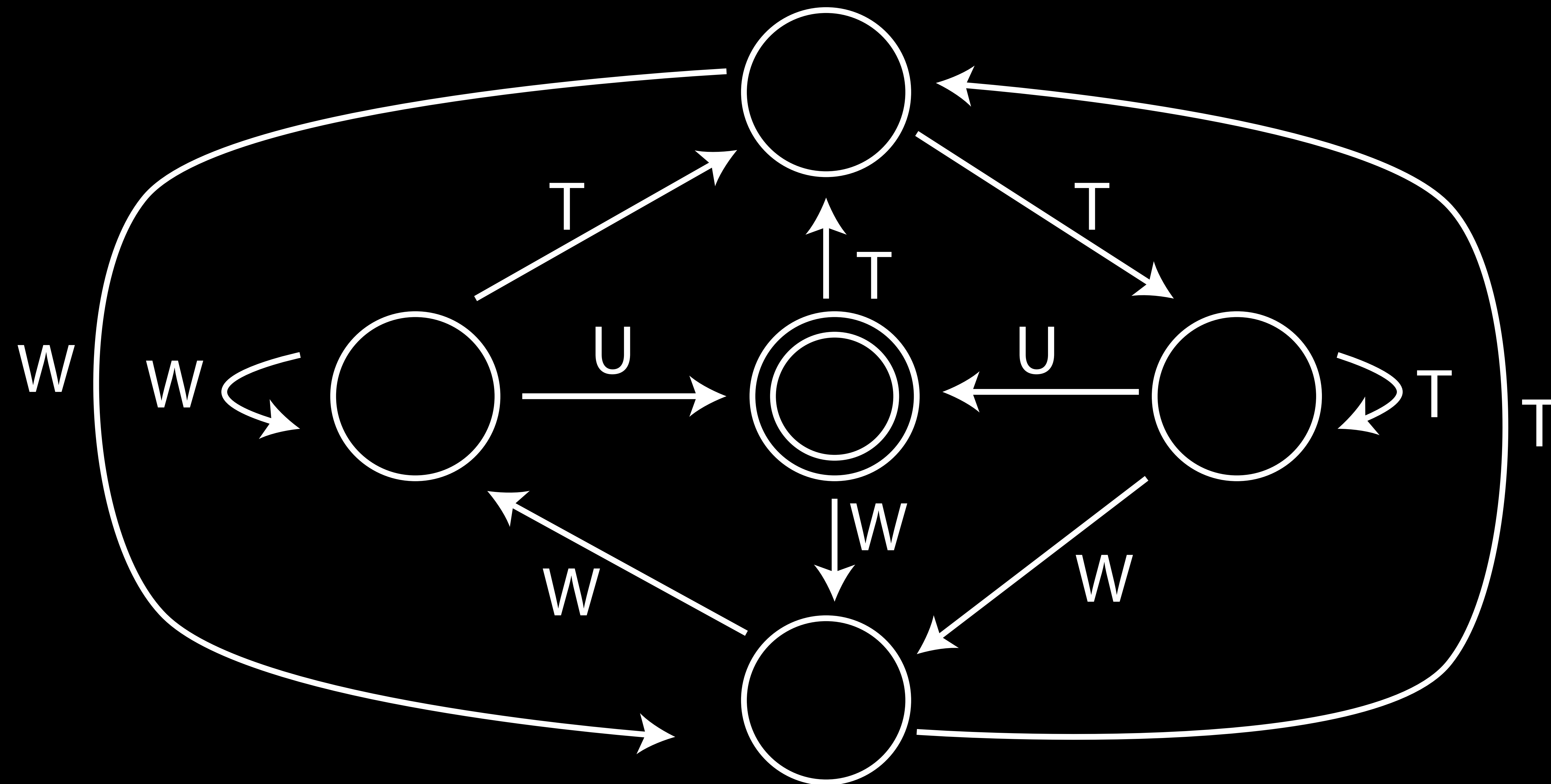


Winding translations and $\mathbb{Z}/3\mathbb{Z}$

- The translation between W/T-sequences and L/R/C-sequences uses mod 3 arithmetic
- Knot end direction given by $[\#W-\#T]_3$
- Validity of tucking underneath a previous strand depends on $[\#W-\#T]_3$ for subsequences
- Examples — convention first move always to L:
  4-in-hand is WTWWU
  Trinity is TWWWTTTUTTU

Rules and conventions for U

- We allow ourselves to write U^k to tuck under the *k*th preceding bow.
- U^k is a valid move if
  - ✳ there are *2k* preceding W/T-symbols
  - ✳ either the first of these *2k* symbols is W and $[\#W\text{-}\#T]_3 = 2$
    or the first of these is T and $[\#W\text{-}\#T]_3 = 1$
    each summed over these *2k* symbols
- For k=1, WWU and TTU are the only valid options



Singly tucked knots are regular

General knots:
either context-free or context-dependent

- Recursive annotated grammar describes the tuck rules:
  thus at most context-dependent and at least context-free
- Our grammar is not immediately amenable to classic pumping lemma arguments

Enumerating

- Winding sequences that start at L and end at R or C are counted by

$$\sum_{\substack{\#w - \#t = 2 \pmod 3 \\ \#w + \#t = k}} \binom{k}{\#t} - 2\binom{k-2}{\#t-1}$$

- Winding sequences that start and end at L are counted by

$$\sum_{\substack{\#w - \#t = 0 \pmod 3 \\ \#w + \#t = k}} \binom{k}{\#t} - 2\binom{k-2}{\#t-1}$$

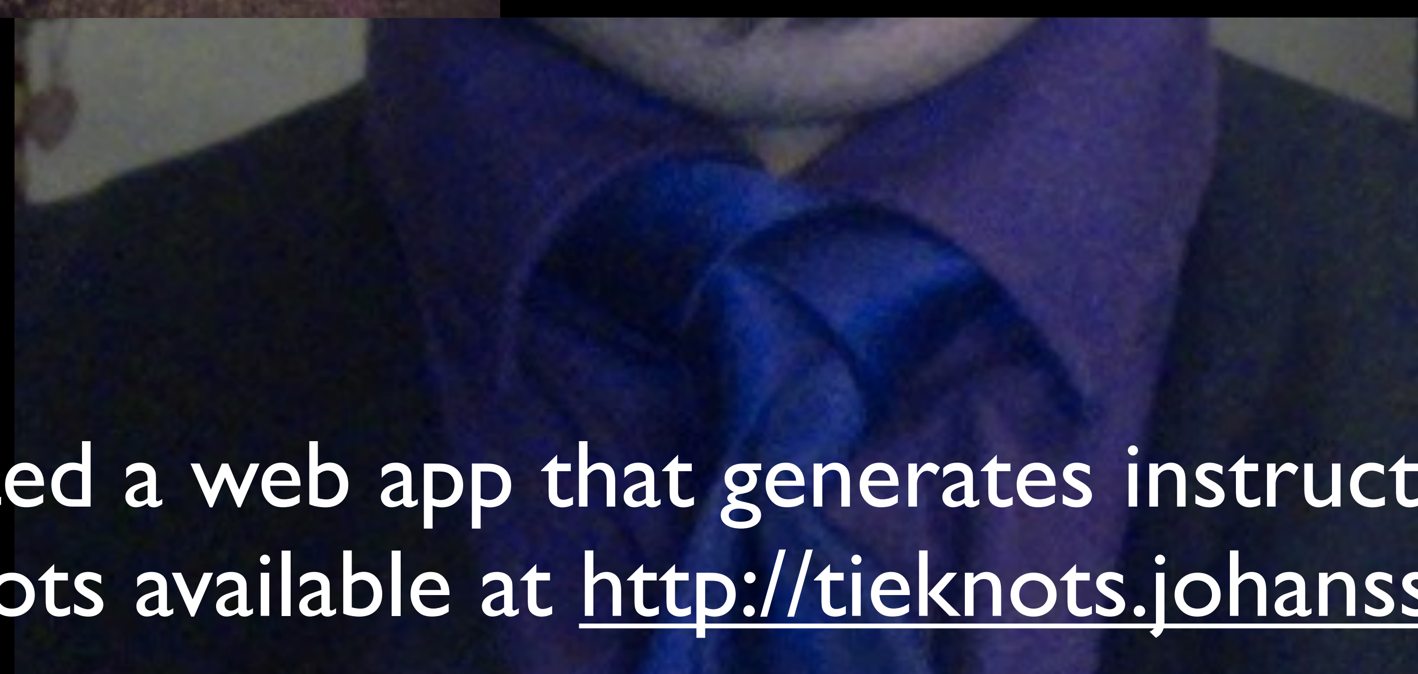
So… how many are there?

Windings

k	2	3	4	5	6	7	8	9	10	11	total
Left	0	2	2	6	10	22	42	86	170	324	682
Center	1	1	3	5	11	21	43	85	171	341	682
Right	1	1	3	5	11	21	43	85	171	341	682

These 2 046 winding patterns generate 177 147 allowable patterns singly tucked knots

Thank you for your attention



We have created a web app that generates instructions for random tie knots available at http://tieknots.johanssons.org