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Towards a topological machine learning

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Overview of existing work

- Stability of persistence
- Persistence of random point clouds
- Distance to measure
- Persistence as ML feature vectors
- Barcode means

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Point cloud cohomology: persistence

- Just computing cohomology: not useful
  Discrete points have no interesting structure.
- Instead:
    1. Cover each point with a ball
    2. Compute cohomology of the union of balls

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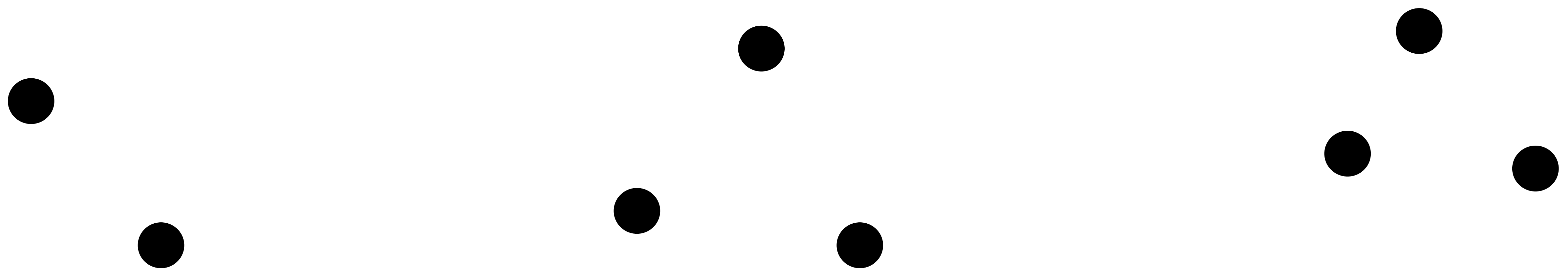
Encoding geometry



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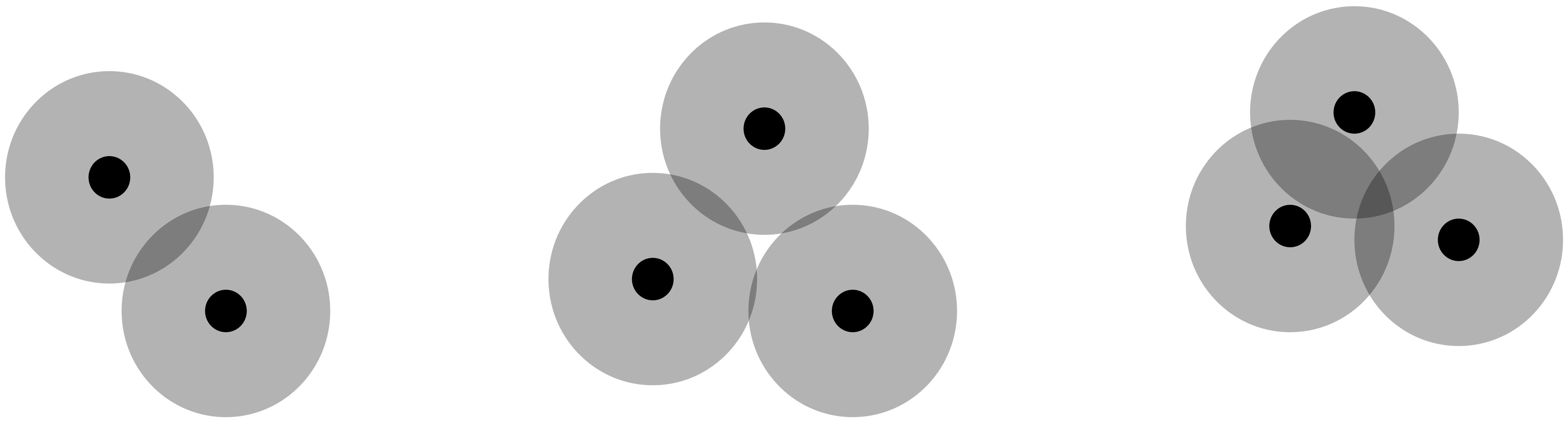

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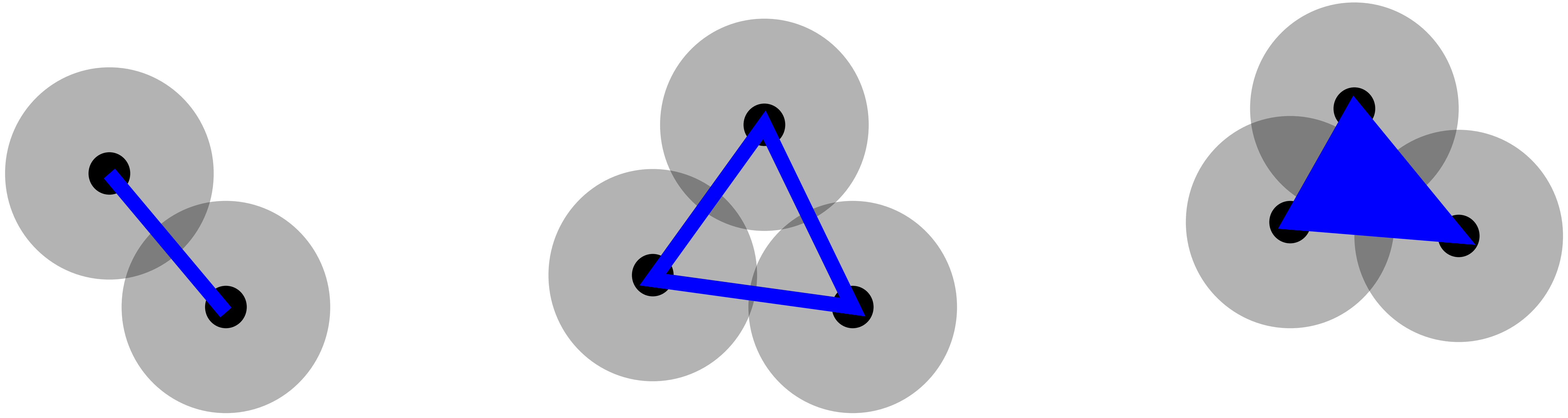

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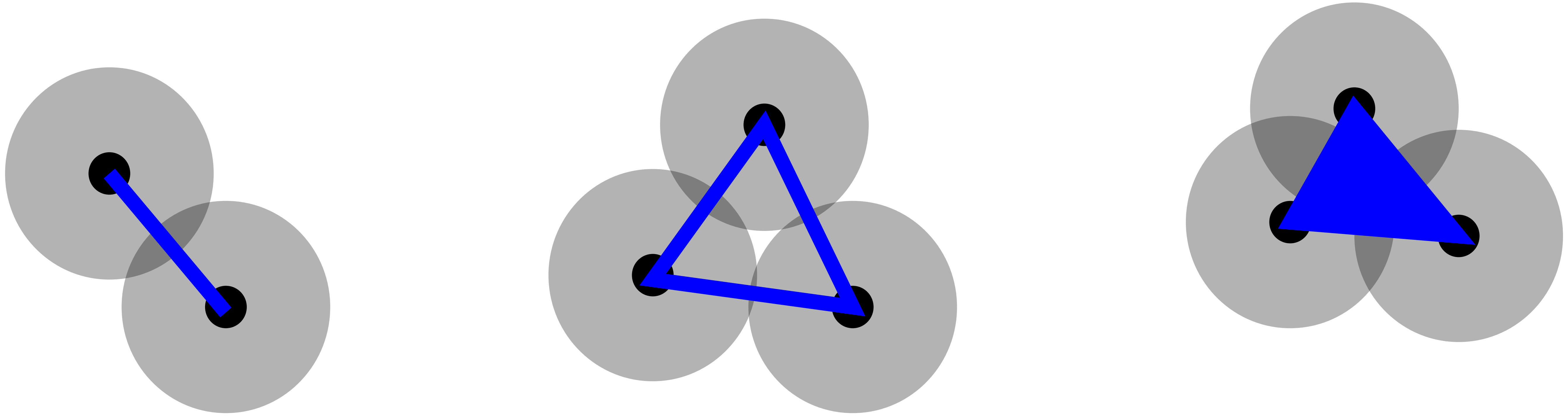

The Čech *complex* of radius r has a simplex for data points $x_0,\ldots,x_n$ whenever all the balls of radius r centered around the data points share an intersection. d_i removes x_i. This generalizes the *single linkage graph*.

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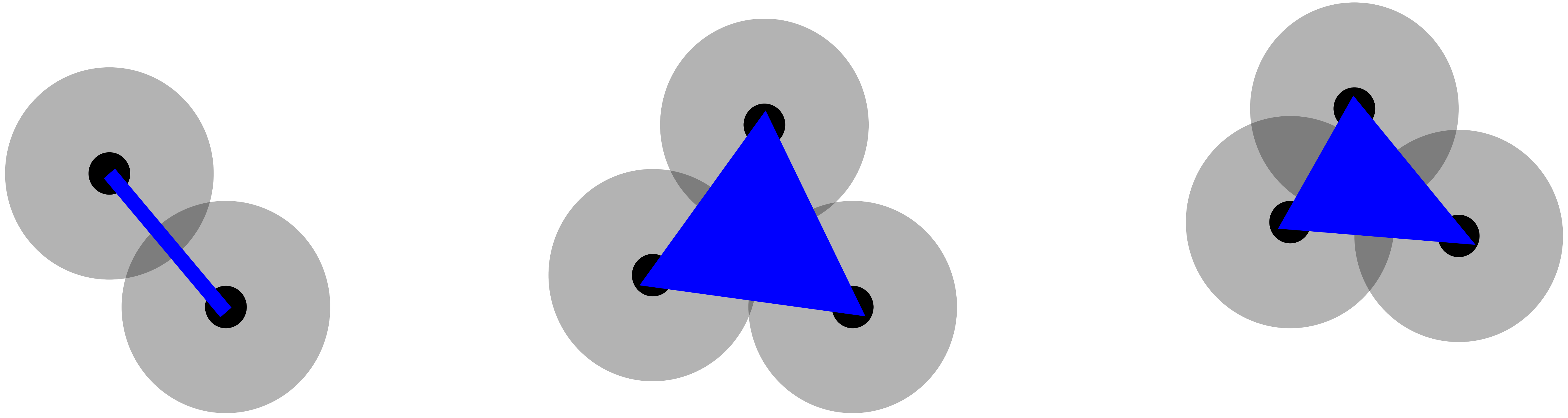

The Vietoris-Rips complex is completely defined by the single linkage graph: includes a simplex whenever there is a clique in the single linkage graph.
Face maps just like in the Čech case.

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Functoriality: algebraic continuity

- If we increase the radius, no cells vanish
  the complex can only ever increase.

  For any $r < r'$ there are inclusion maps
  $\check{C}_r(X) \to \check{C}_{r'}(X)$

- Computing cohomology is a functor — continuous in an algebraic sense:
  if there is a map $X \to Y$, then there is an induced map
  $H^1Y \to H^1X$

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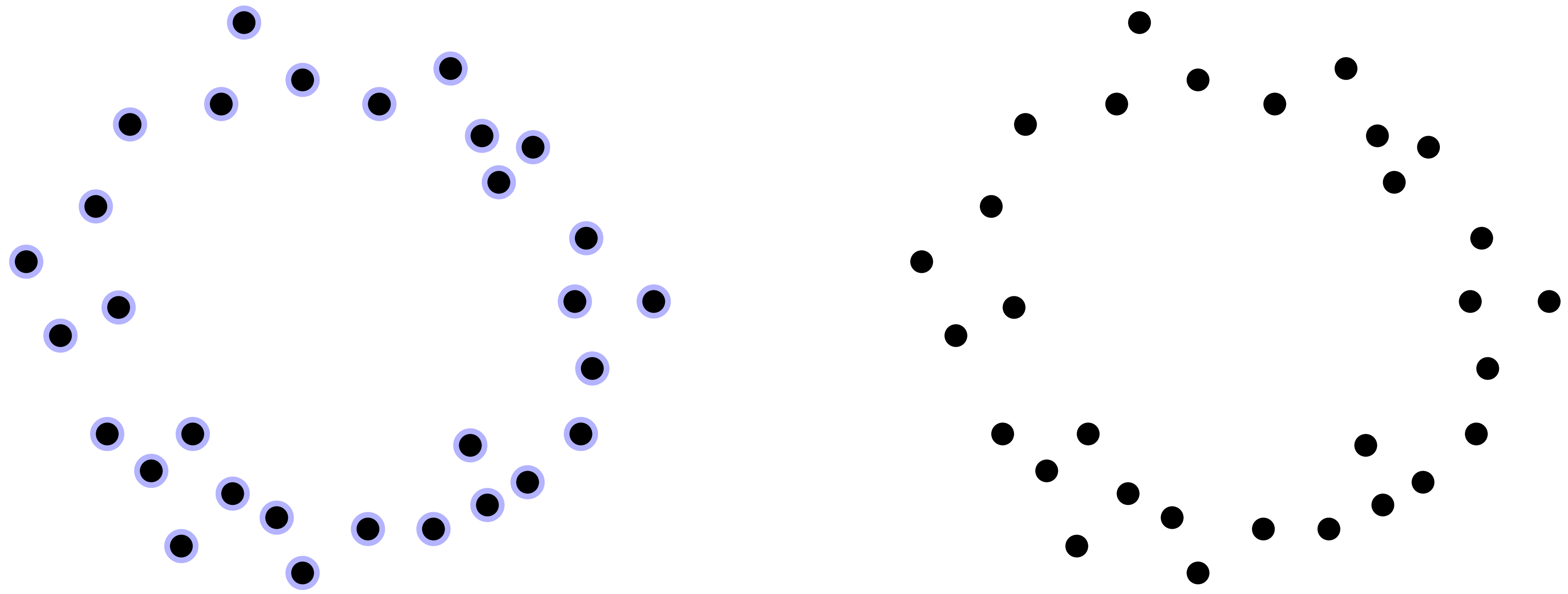
Persistent cohomology: tracking maps



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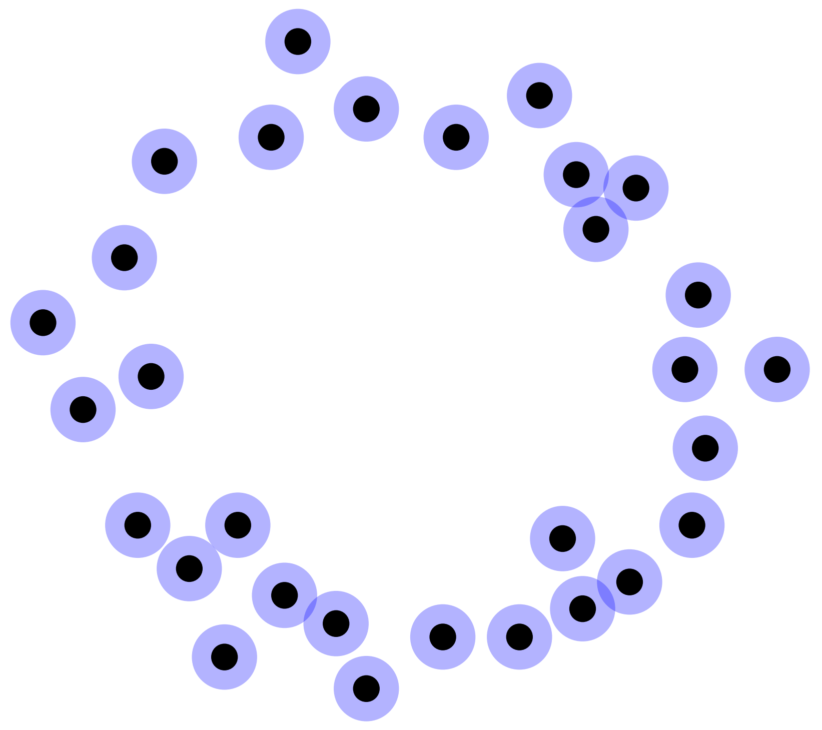
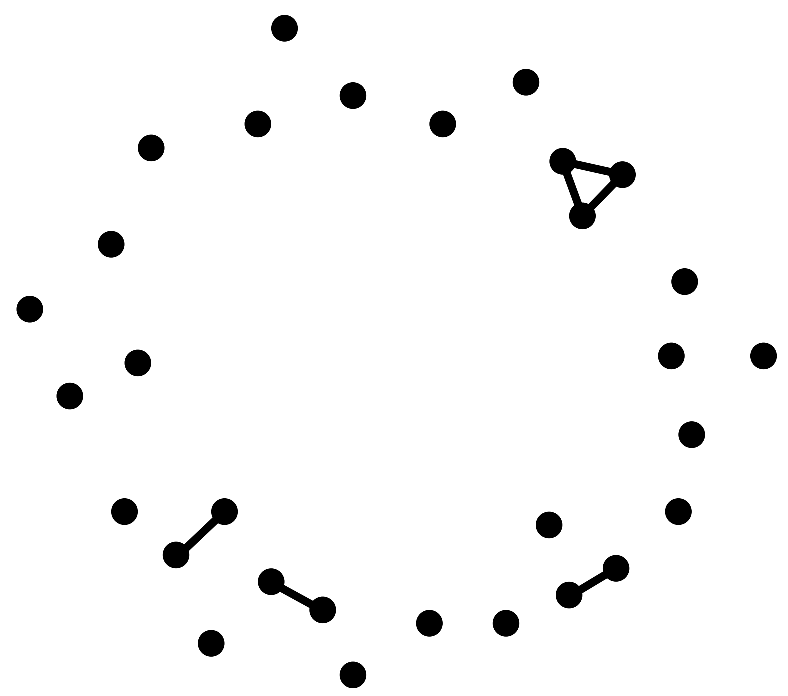
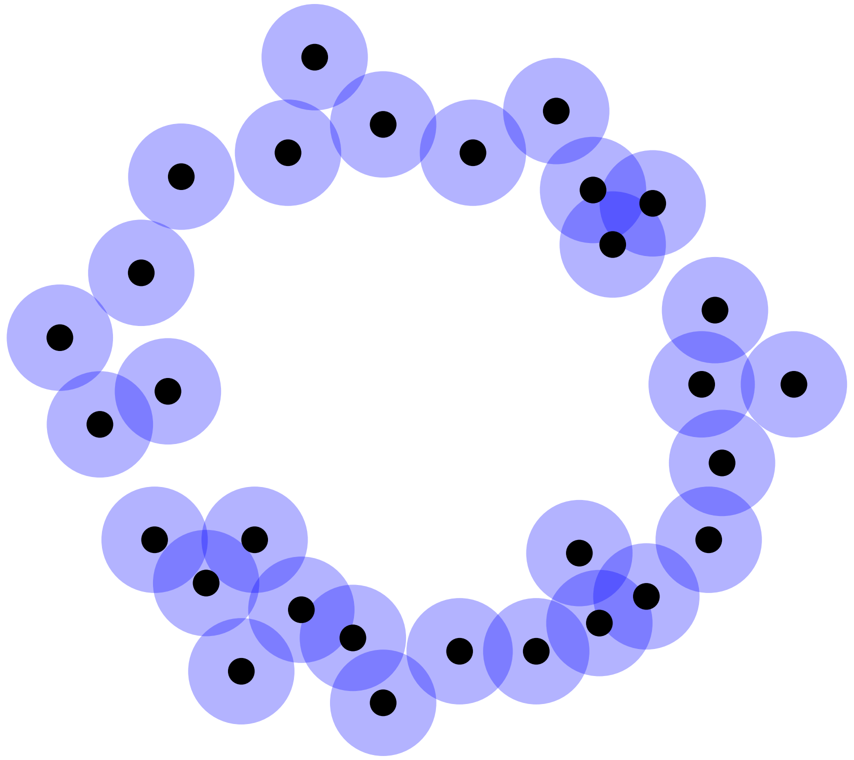
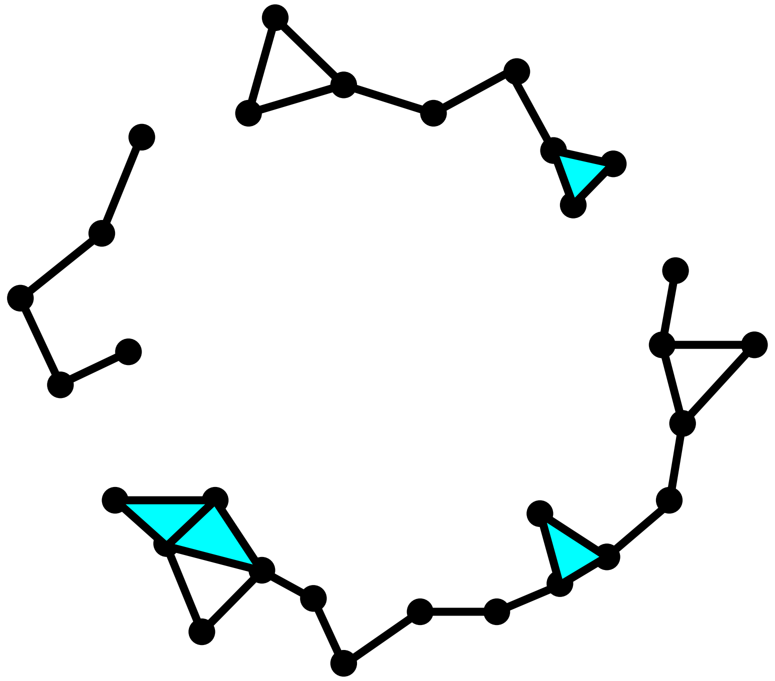




1 loop

Persistent cohomology: tracking maps





3 loops

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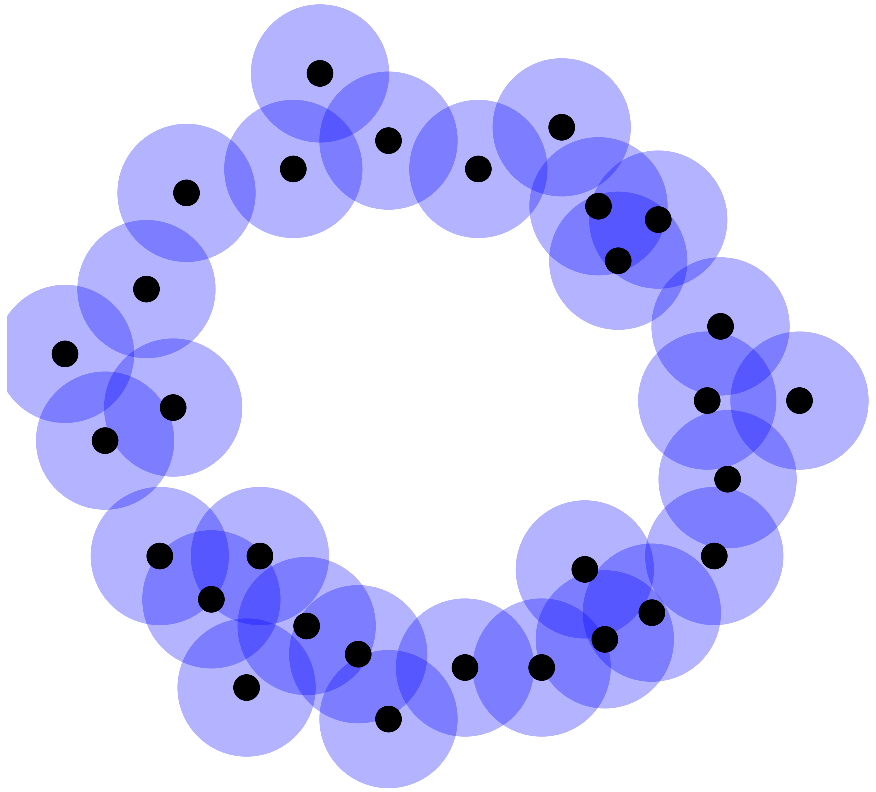
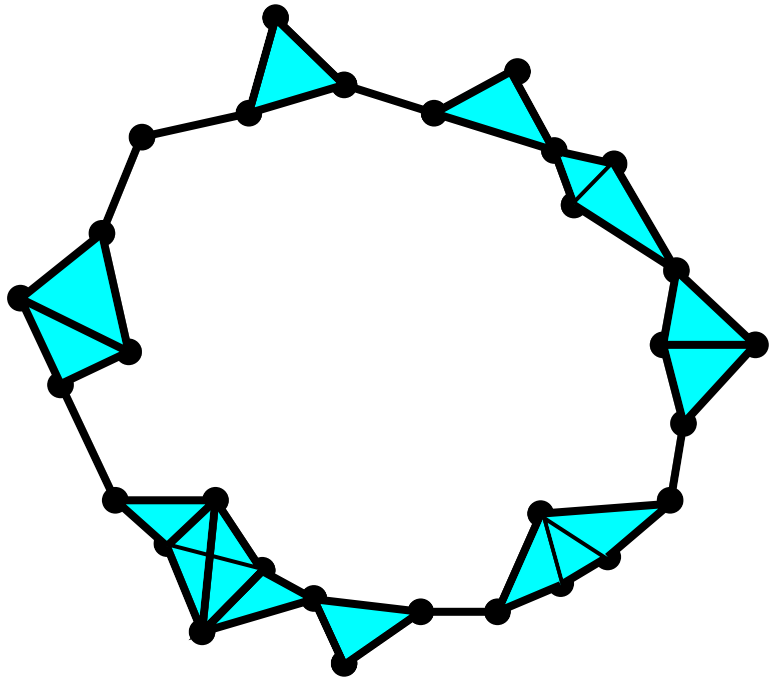
Persistent cohomology: tracking maps





1 loop

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Key point:
are loops the same?

- The induced map come to our rescue:
- From a sequence of spaces

  $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots \hookrightarrow X_n$

  cohomology produces a sequence of vector spaces

  $H^1X_n \to \ldots \to H^1X_3 \to H^1X_2 \to H^1X_1$

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Algebra glues features

- Theorem (Gabriel, 1972): If M is a collection of vector spaces with linear maps along a path, M decomposes into interval modules.
- Decomposition produces birth/death pairs just like our barcodes.

Interval decompositions produce salient features

- Decomposing a diagram such as

  $H^1X_n \to \ldots \to H^1X_3 \to H^1X_2 \to H^1X_1$

  produces a direct sum diagram where each summand has the shape

  $0 \to \ldots \to 0 \to \mathbb{k} \to \ldots \to \mathbb{k} \to 0 \to 0 \to 0$

  where all maps $0 \to 0$ and $\mathbb{k} \to \mathbb{k}$ are isomorphisms.

- Each such summand is a choice of a (higher dimensional) essential circle coordinate across parameter values that persists across different values.

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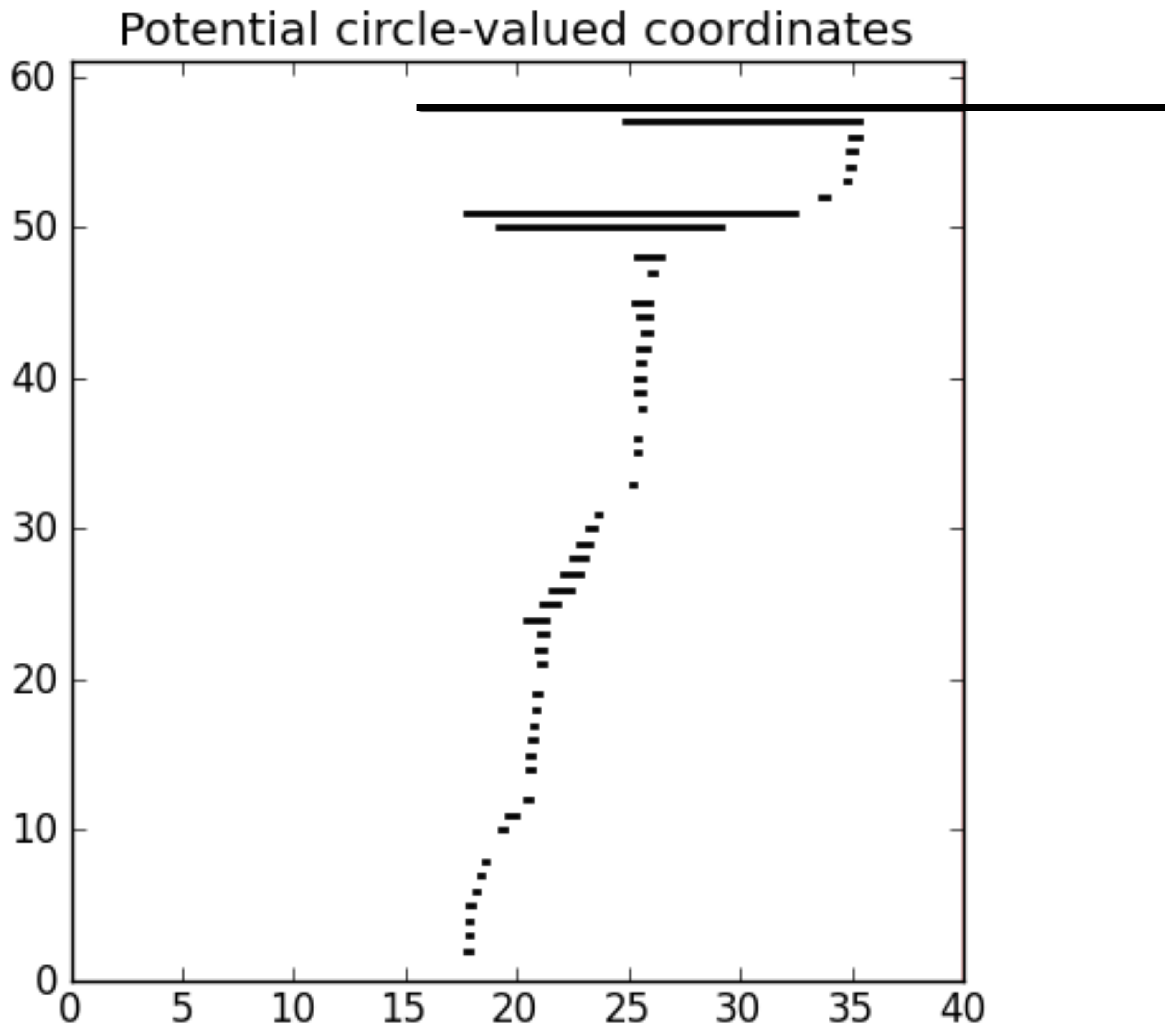
Barcodes and persistence diagrams

- Visualization tools for topological information
- Displays each interval in the decomposition



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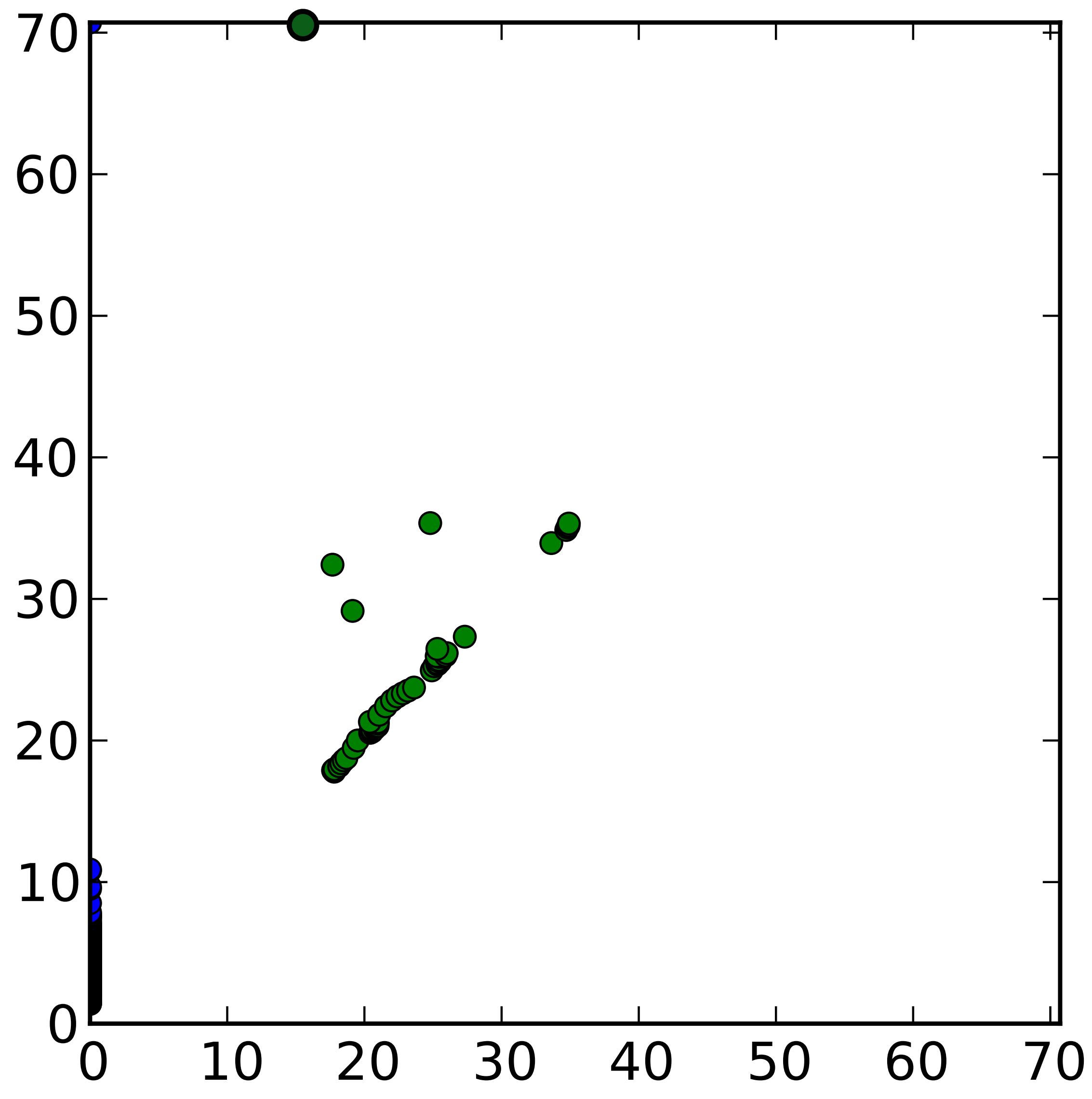

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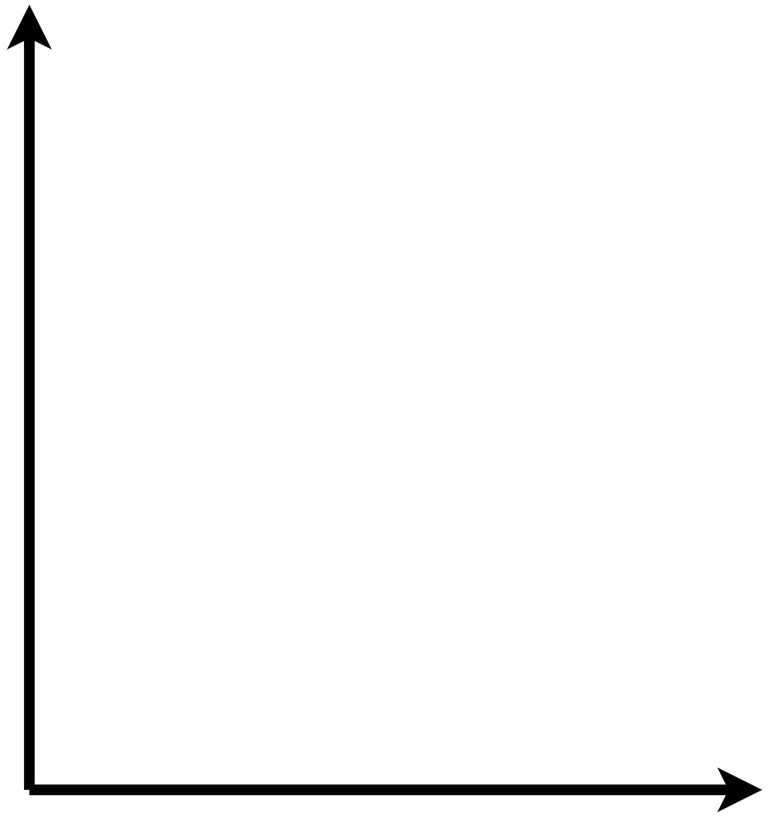
Barcodes are stable



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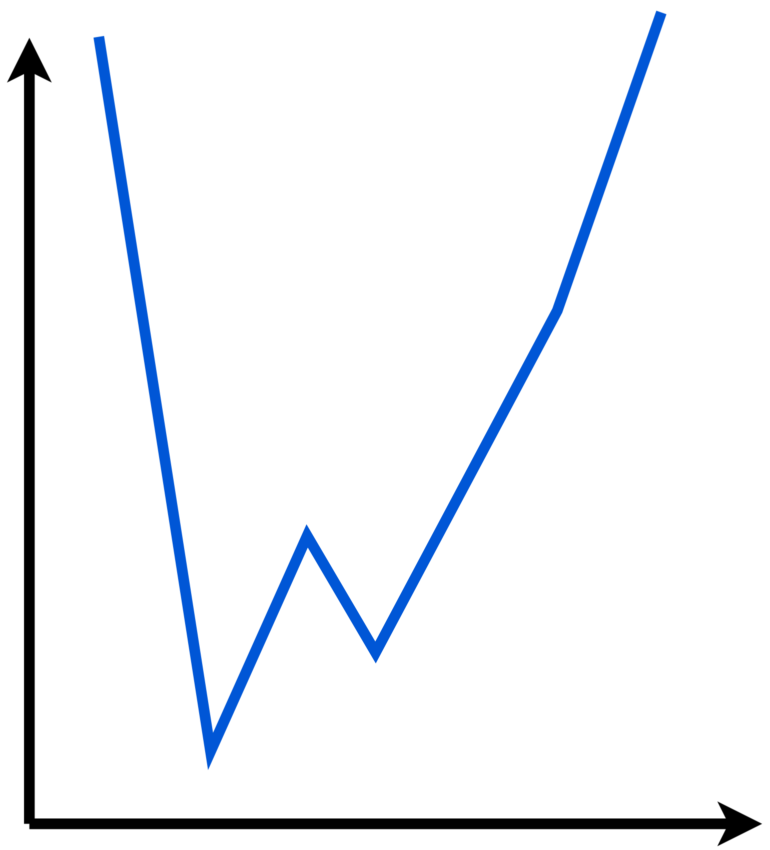

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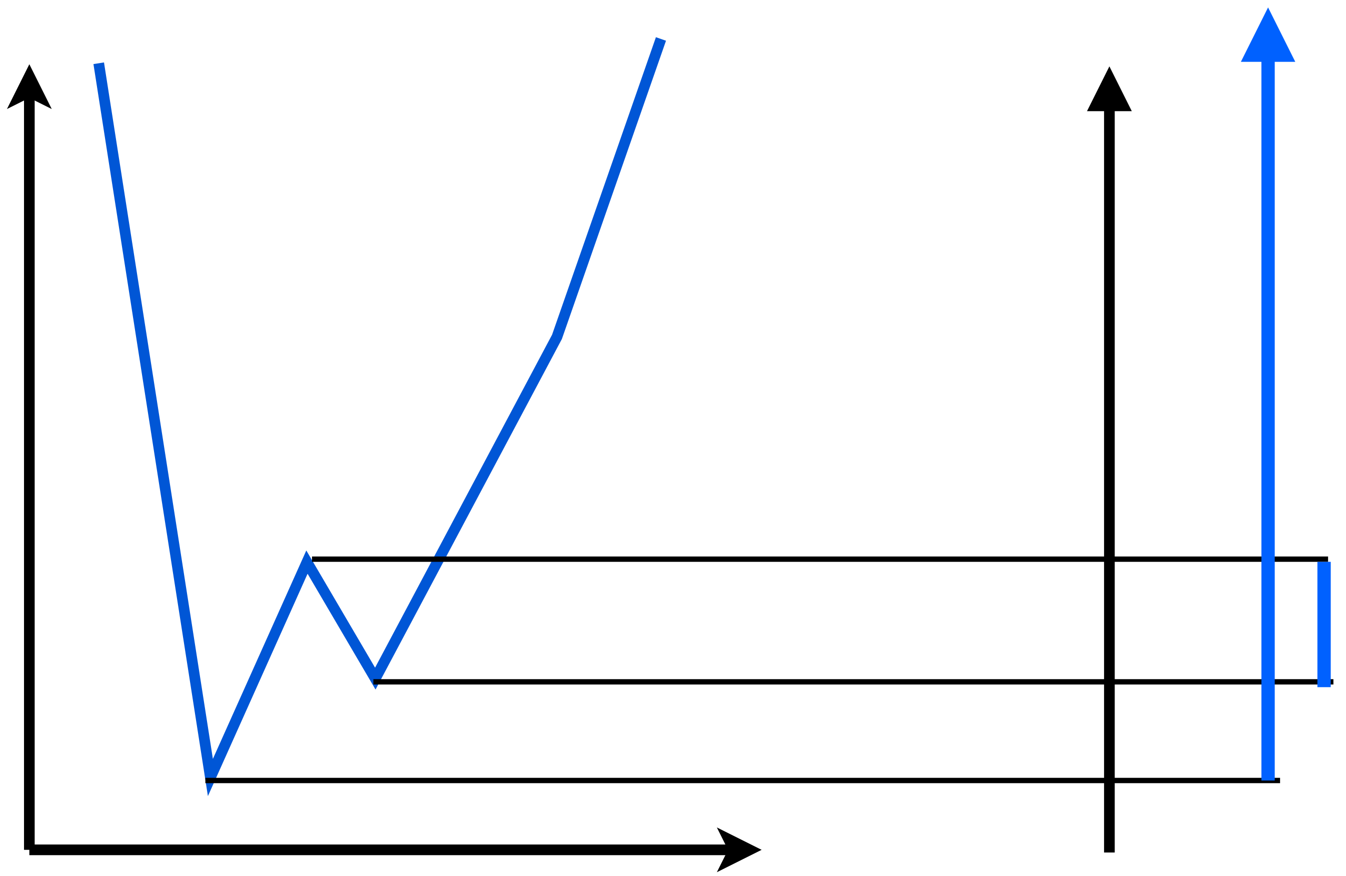

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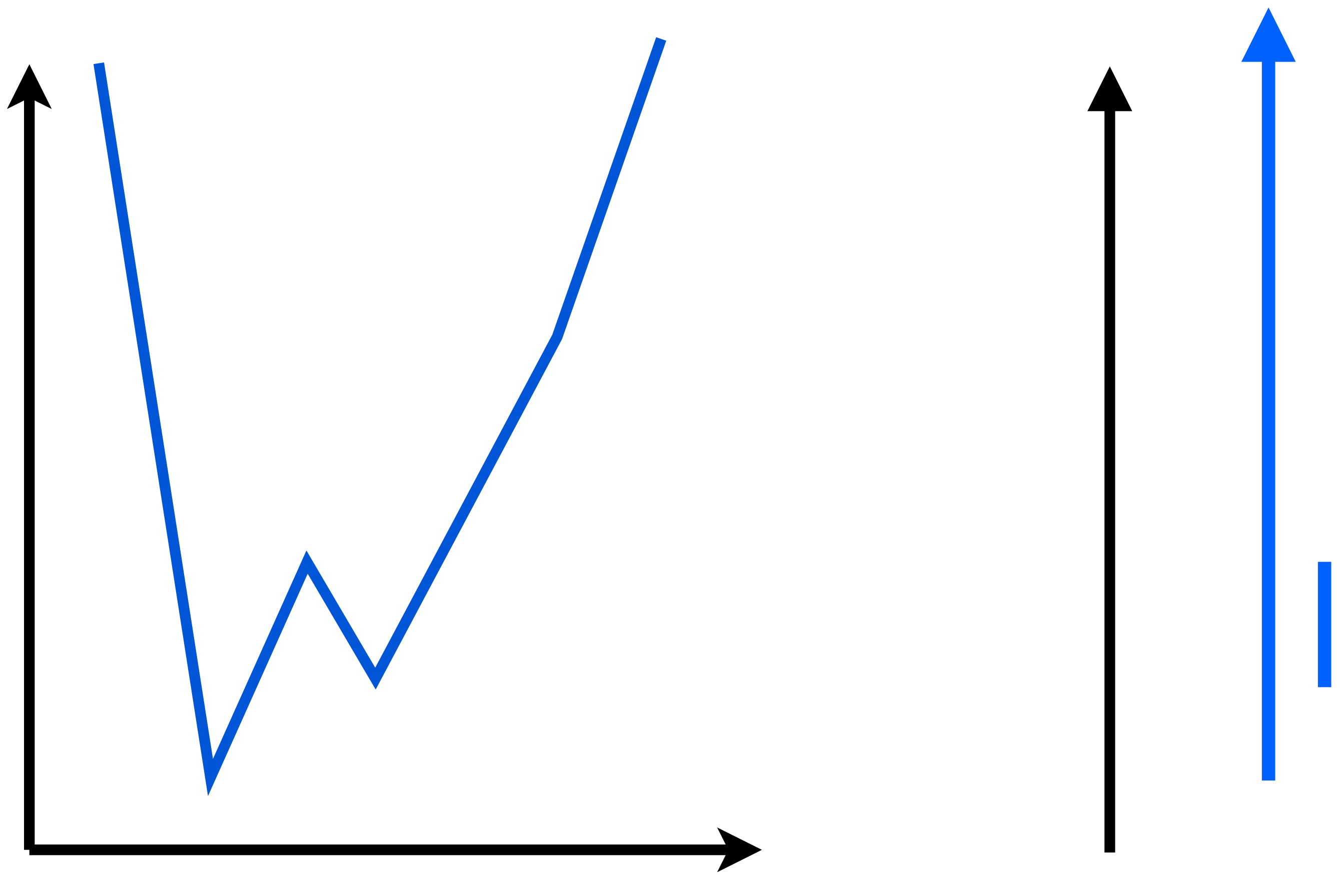

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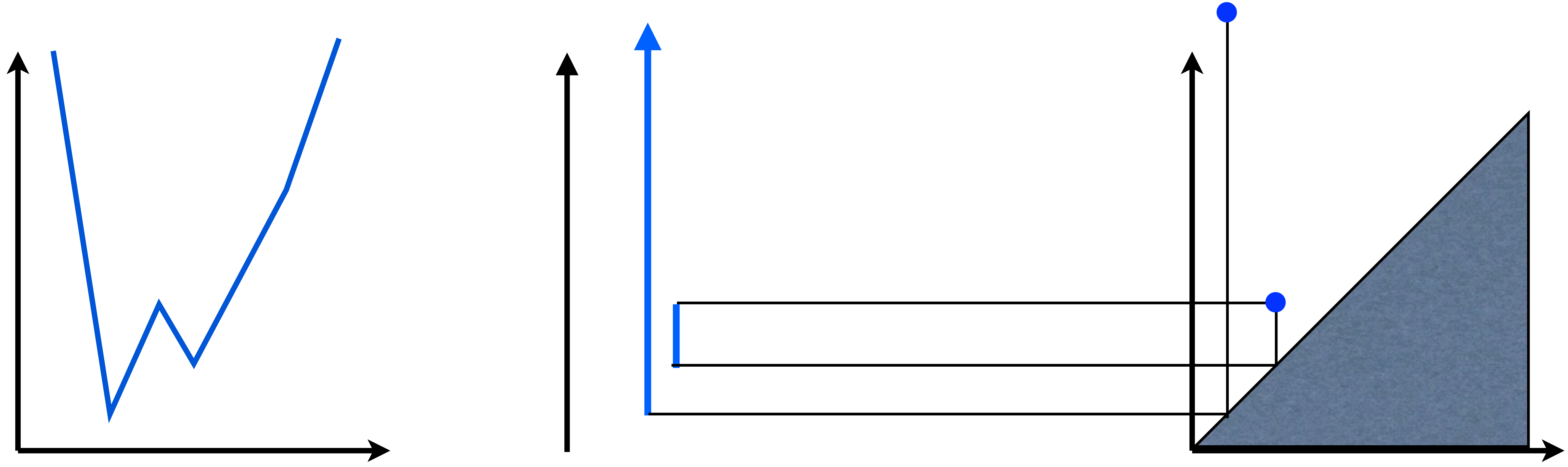

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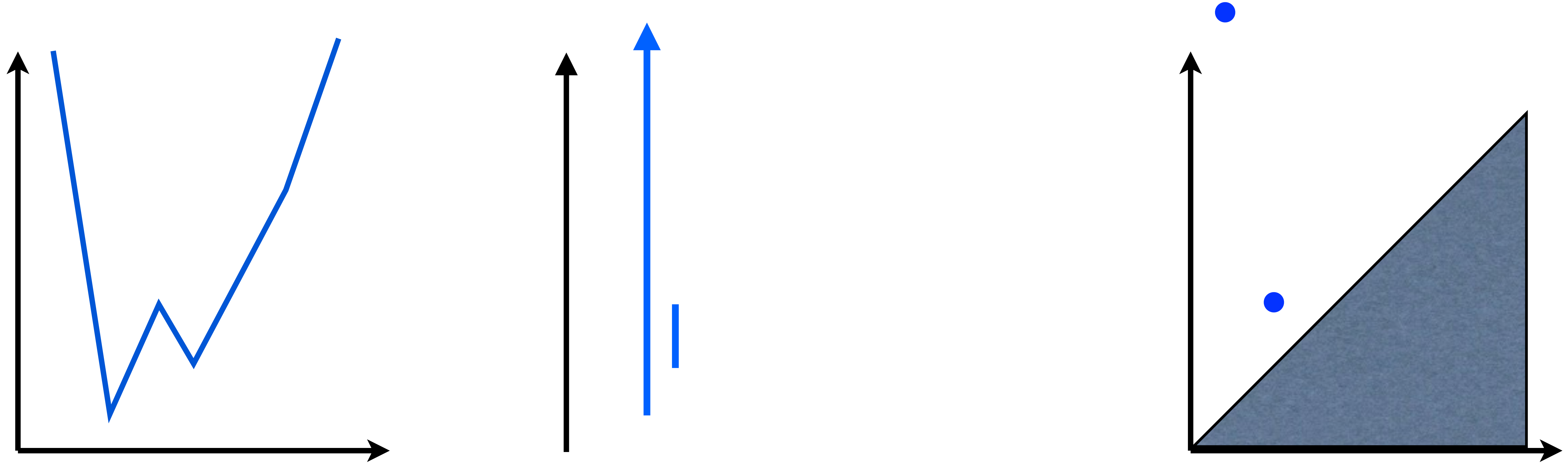

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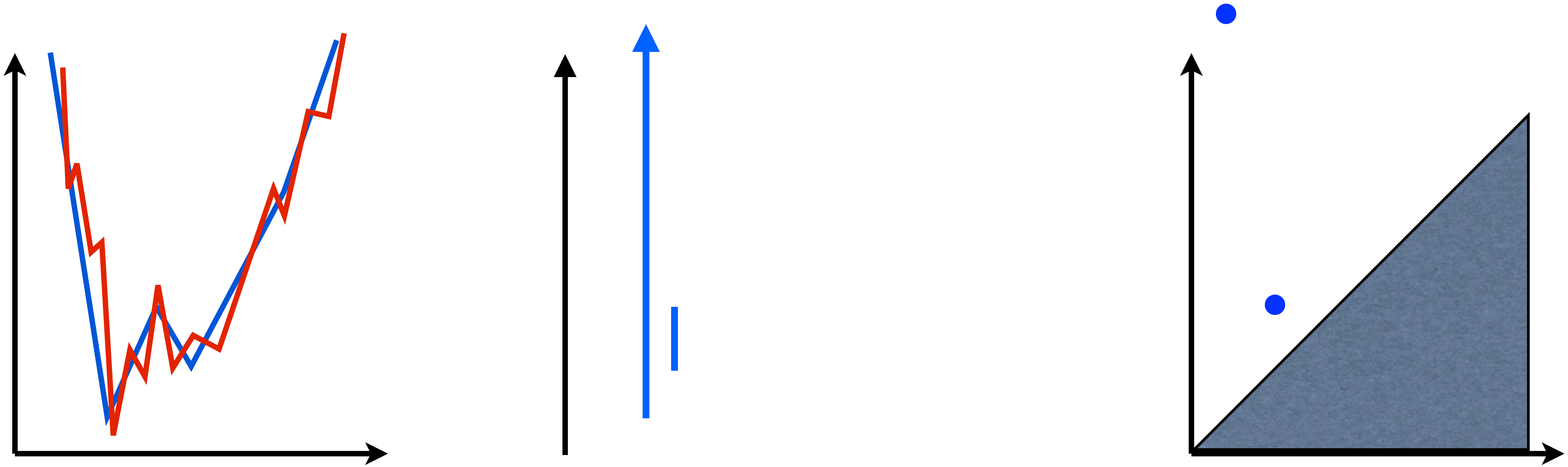

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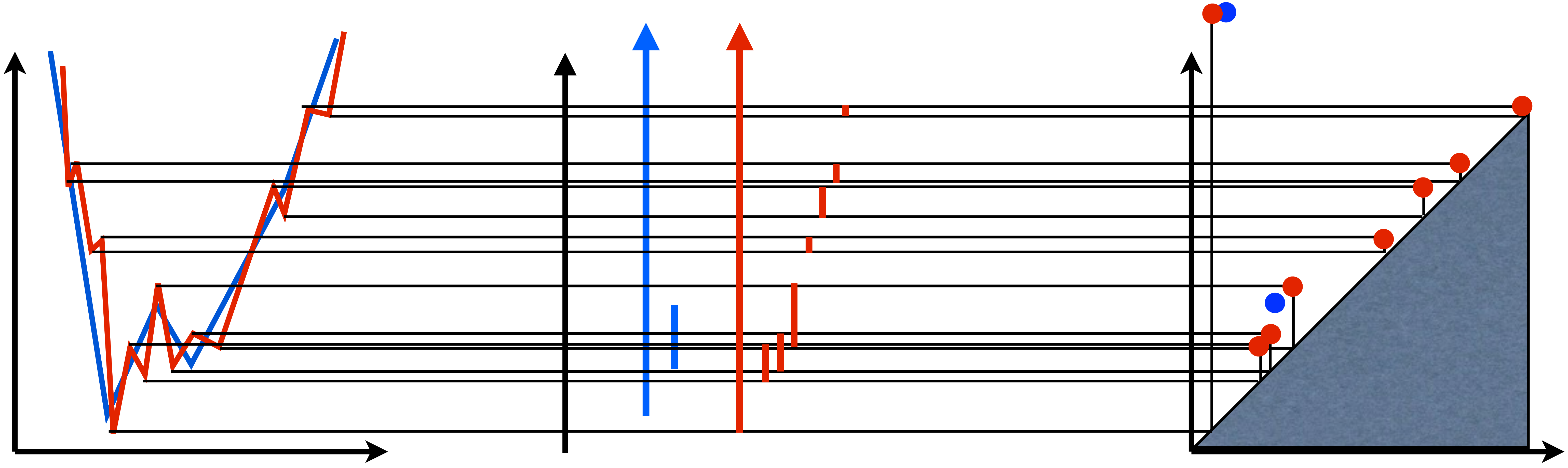

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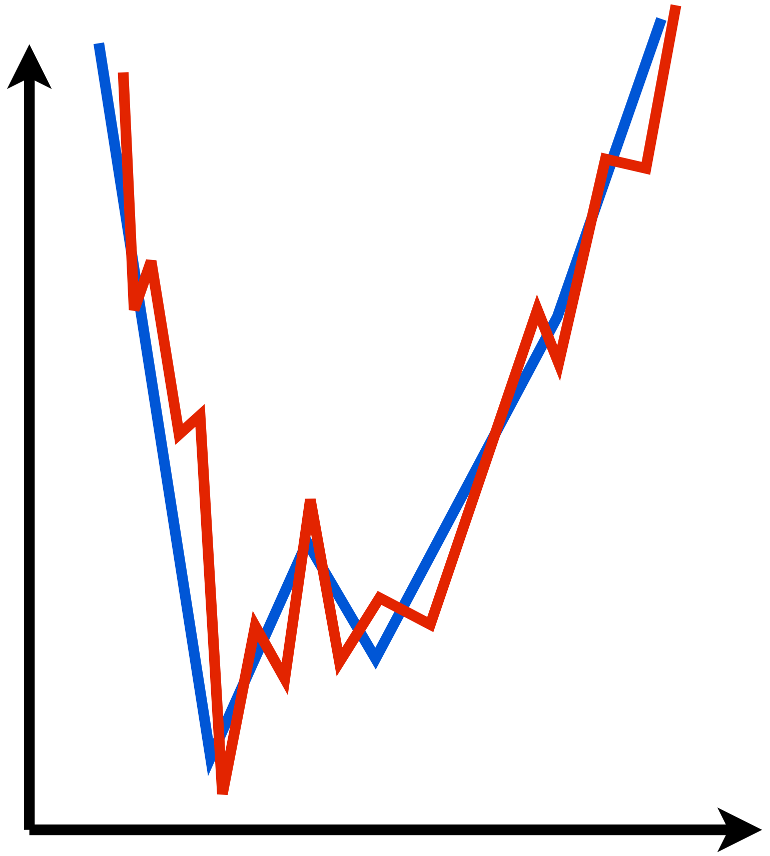

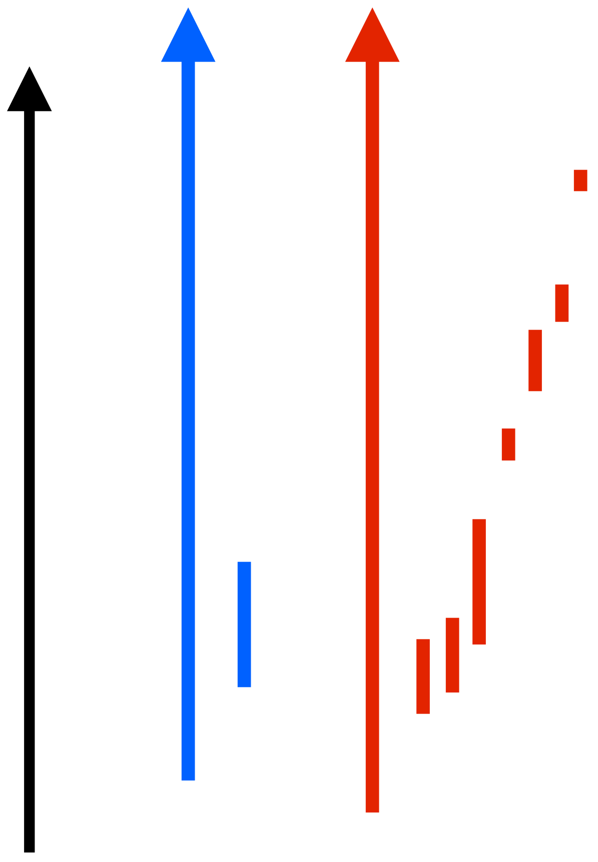

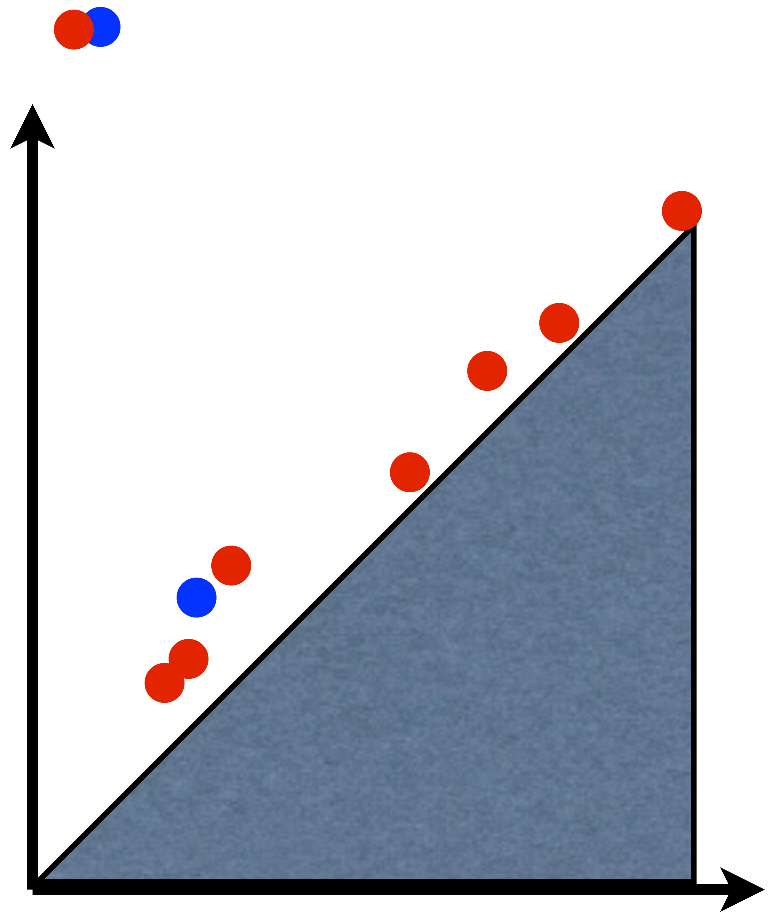

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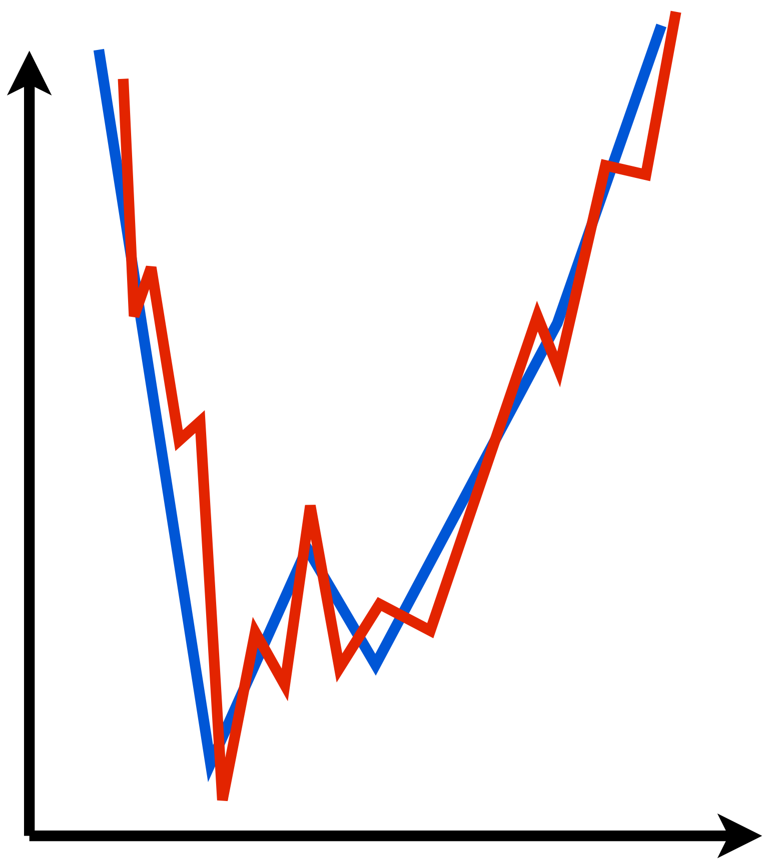
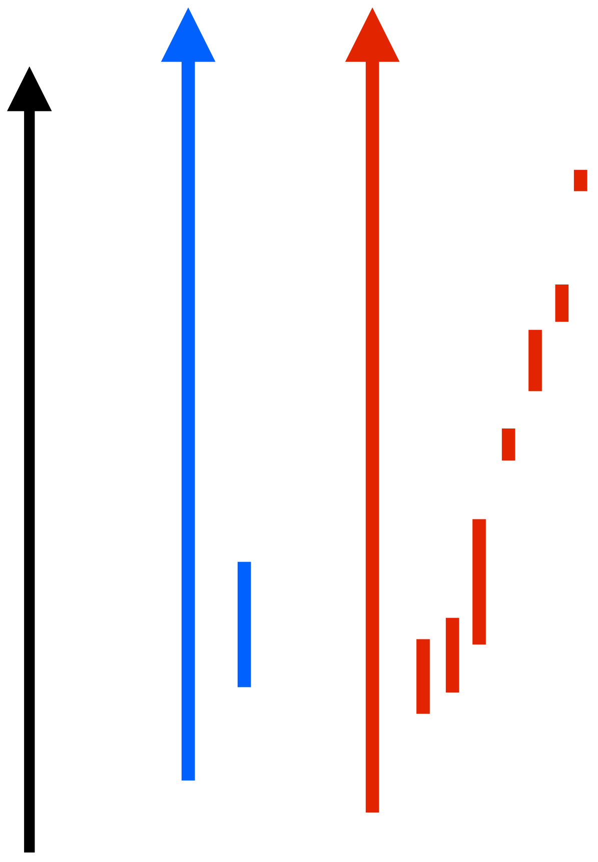
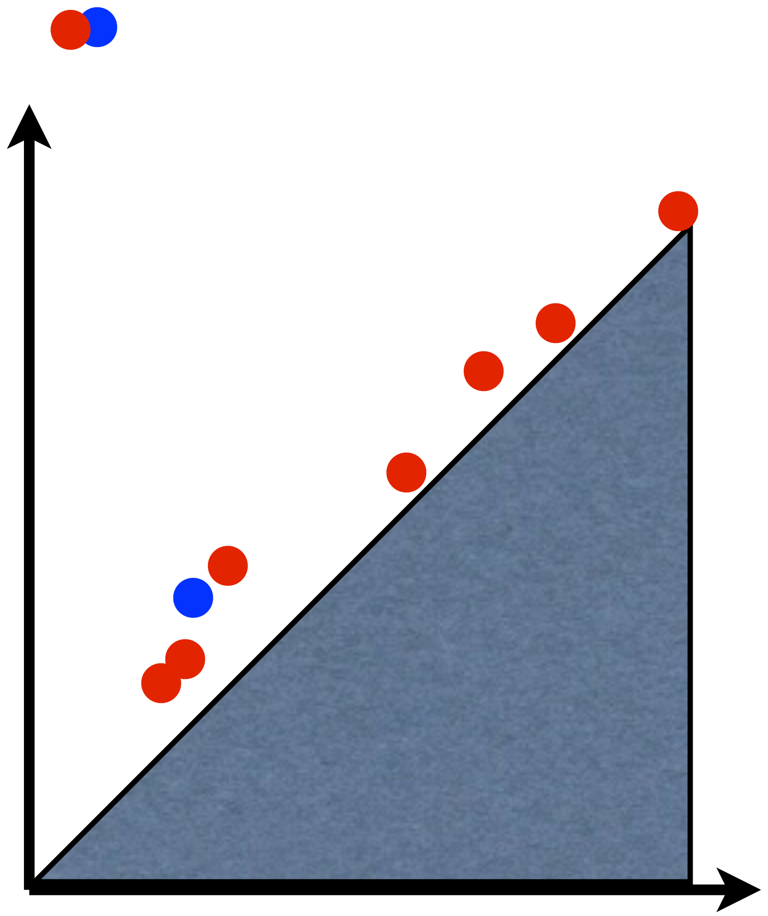


L_∞ distance of functions





upper bound on maximal distance between points

upper bound on Wasserstein distance

isometric to *interleaving distance*

Stability of persistence

- Persistence is:
  the evolution of the topology of sublevel sets of a function on a (sample from a) manifold.
- Stability of persistence:
  the mapping from the function-and-manifold to the persistence diagram is
  - Continuous
  - 1-Lipschitz
- Therefore: what persistence computes is related to the original topology. Small features go away with small perturbations.

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Persistence of random points

- Zomorodian:
  Random flag complex. β_n and β_{n+2} (mostly) don't overlap
- Adler et.al.:
  Excursion sets of random fields: Snap, crackle, pop.
- Kahle:
  Phase transitions for persistence of random points in $\mathbb{R}^d$ for a wide family of probability distributions.

Distance to measure

- Consistent, stable and easy to compute estimator of topological features of an underlying probability distribution.
- Connect points to their nearest neighbors. Work with the resulting filtered simplicial complex.
- Chazal, Guibas, Oudot, Skraba (2013)

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Machine learning features

- Adcock, Carlsson, Carlsson:
  Classify functions on barcodes to make it easier to pick features
- Berwald, Gidea, V-J:
  Classify different modes of dynamics using ML on persistence barcodes
- Topology as dimensionality reduction.
- Bubenik: Persistence landscapes

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Mean barcode Confidence intervals

- Fasy, Lecci, Rinaldo, Wasserman, Balakrishnan, Singh: Confidence sets for persistence diagrams
- Turner, Mileyko, Mukherjee, Harer: Fréchet means for distributions of persistence diagrams
- Munch, Bendich, Turner, Mukherjee, Mattingly, Harer: Probabilistic Fréchet means and statistics on vineyards
- Mileyko, Mukherjee, Harer: Probability measures on the space of persistence diagrams

New directions

- Machine learning provides tools that can make persistent homology work better.
- I will describe my latest research ideas next: using density estimation to build streaming topological learning.

Streaming persistence

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- Inspiring example:
  - Tape accelerometer to a shoe.
  - Measure signal:
    different closed curves for different steps/gaits.
  - Build a topological model of this recurrence.
- Problem:
  - Want to run data capture for a long time.
  - Persistence behaves badly [$O(n^6)$] with size.
  - 100 runs with 100 points faster than one run with 10 000 points.
- We need to control input size to persistence.

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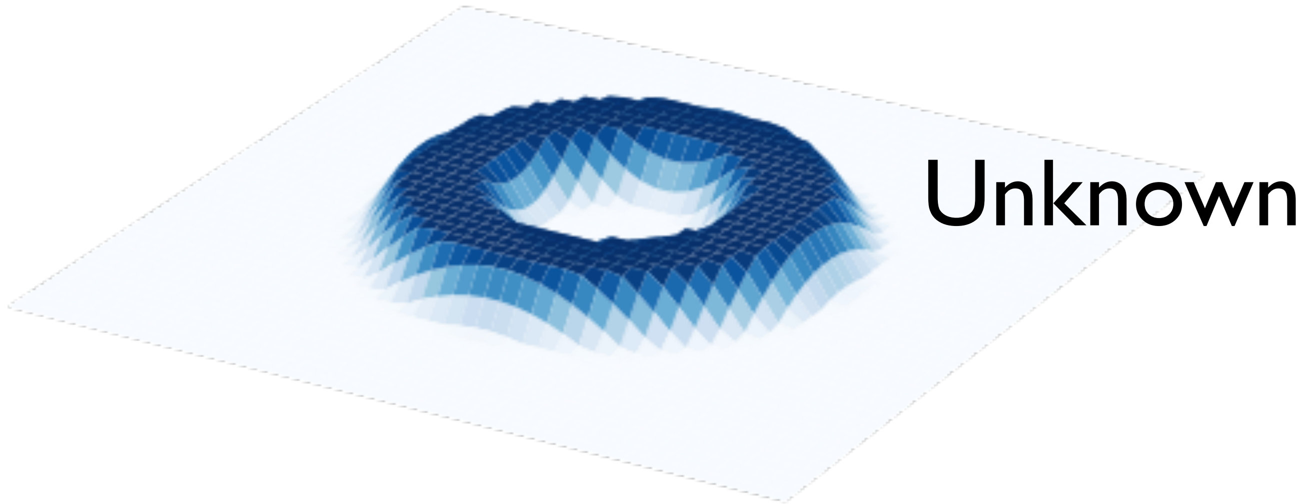
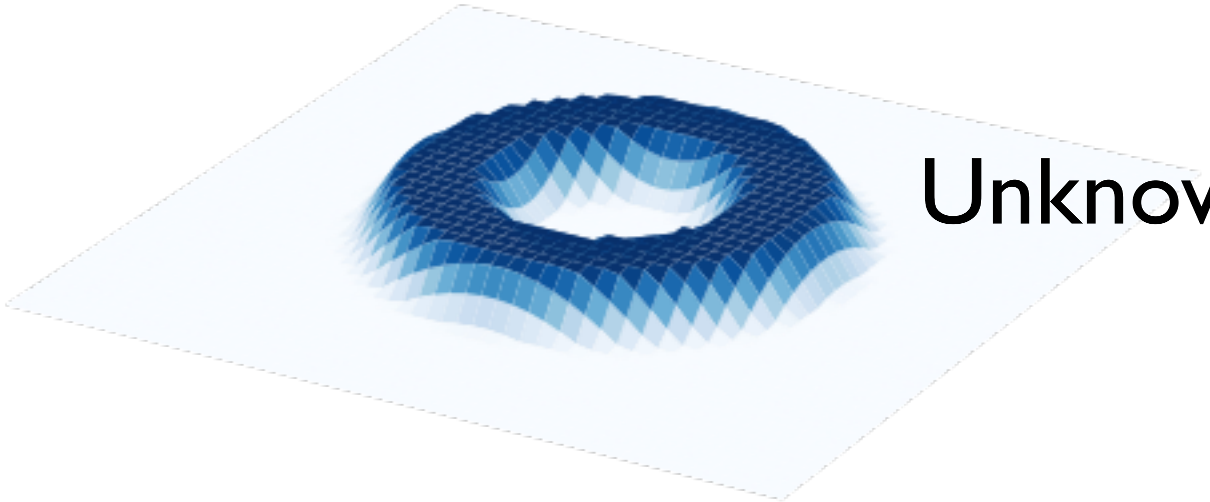
Streaming persistence



Streaming persistence

Very large
Appears gradually



Unknown

Streaming persistence

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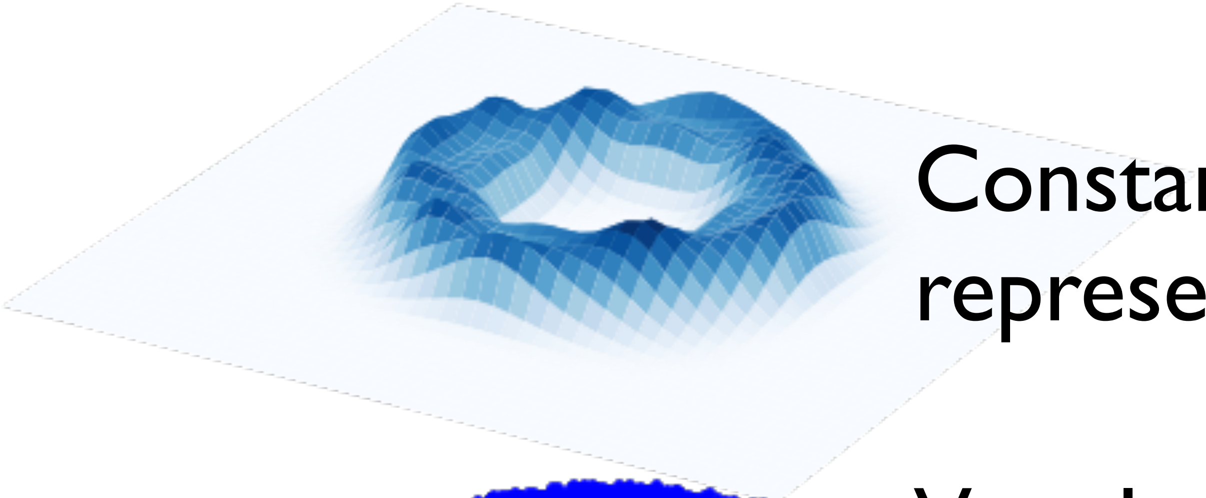

Constant size
representation

Very large
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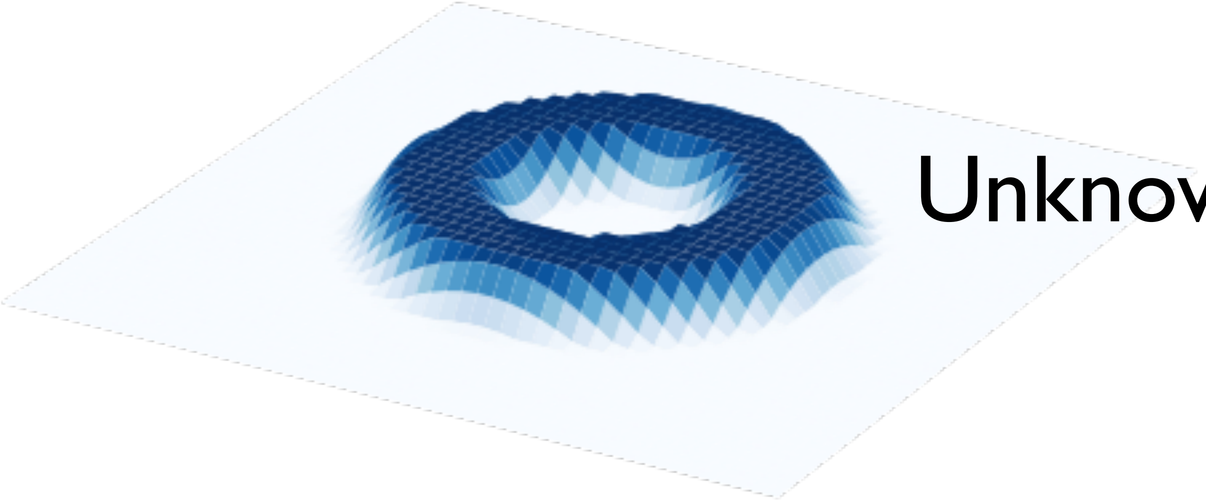
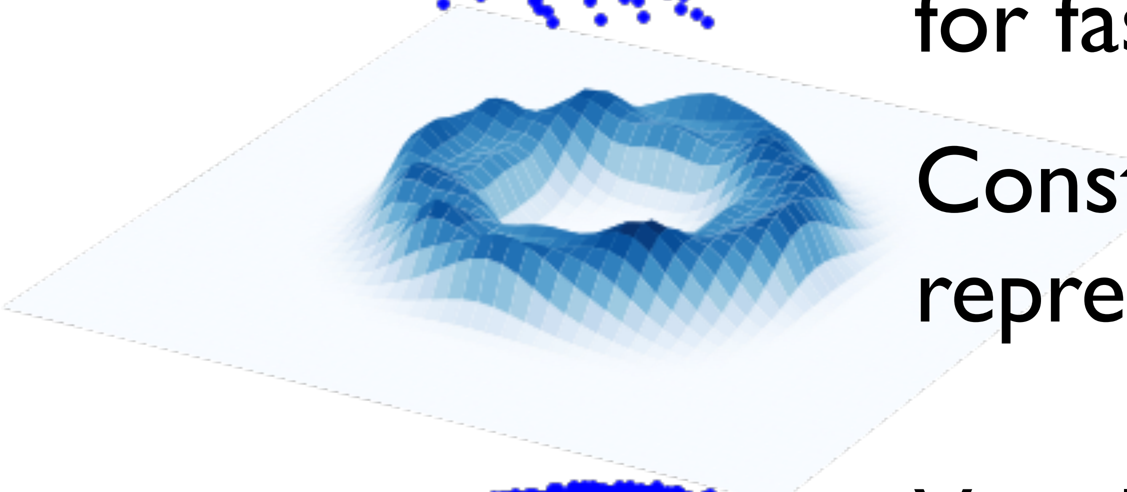


Unknown

Streaming persistence

Small enough
for fast persistence



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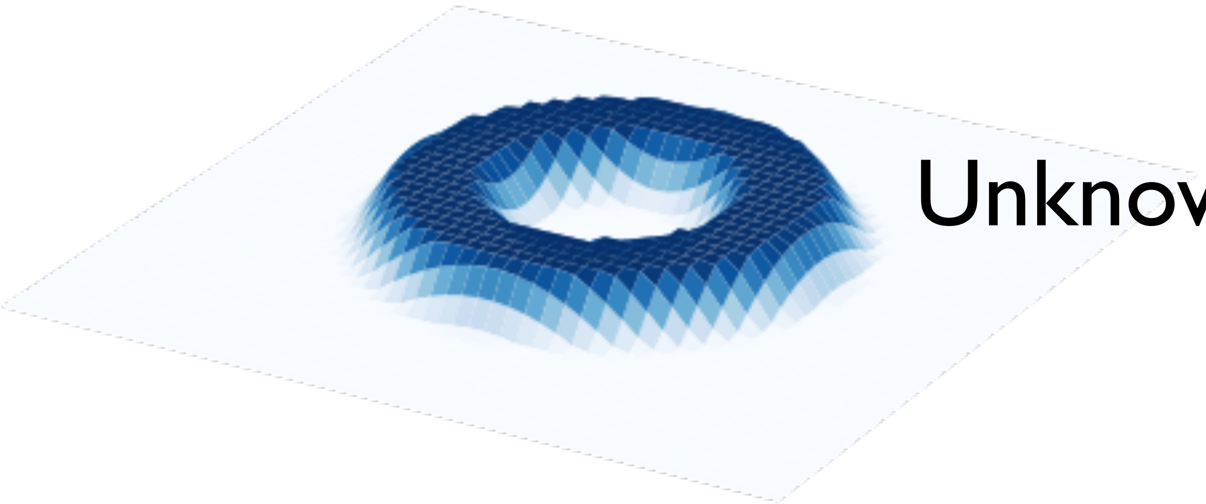

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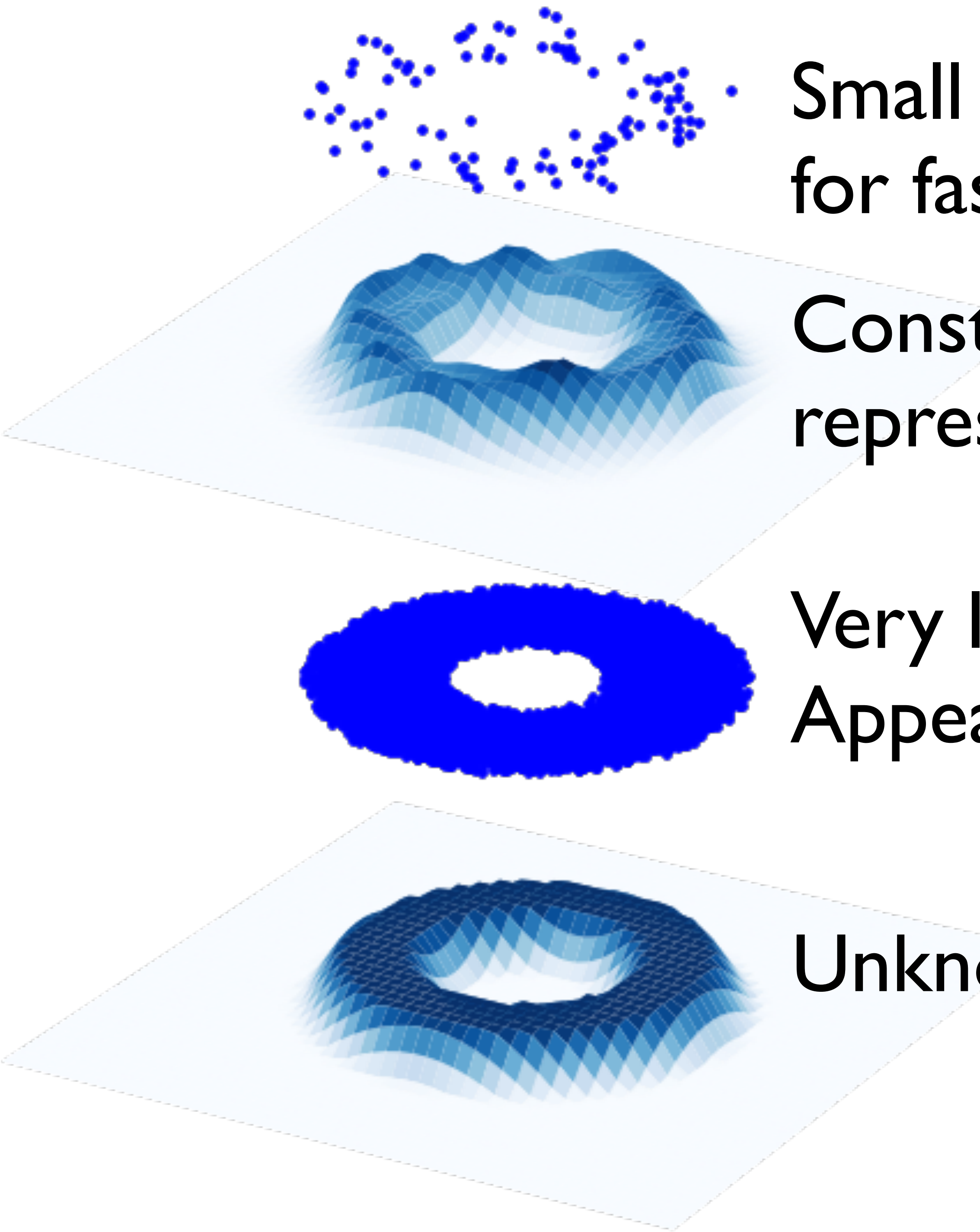

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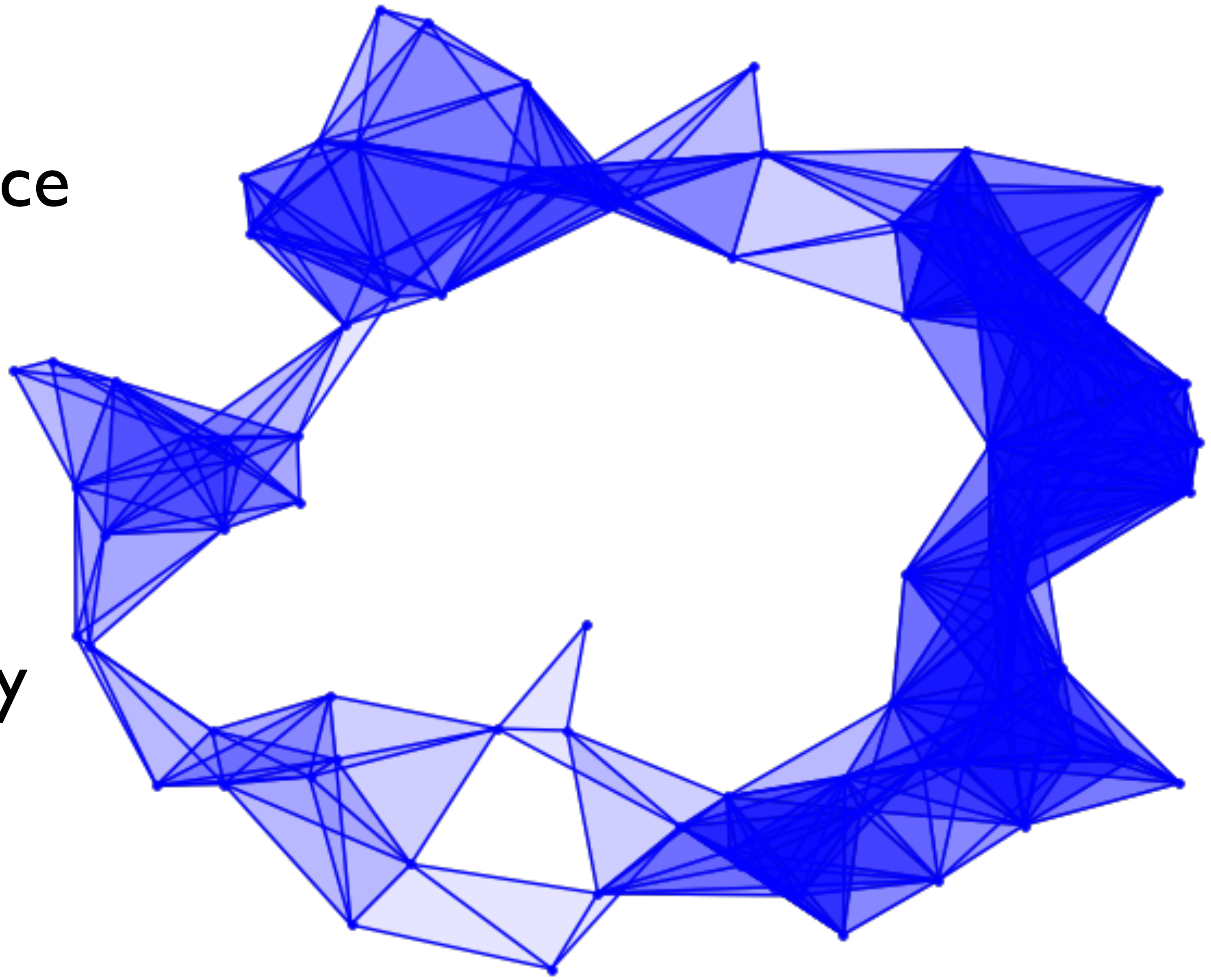

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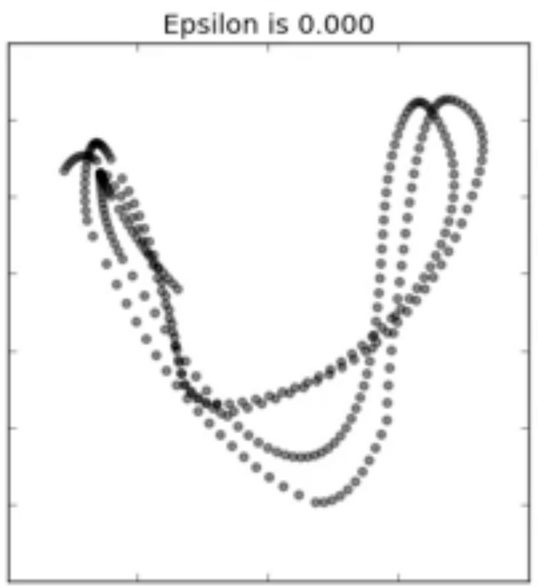
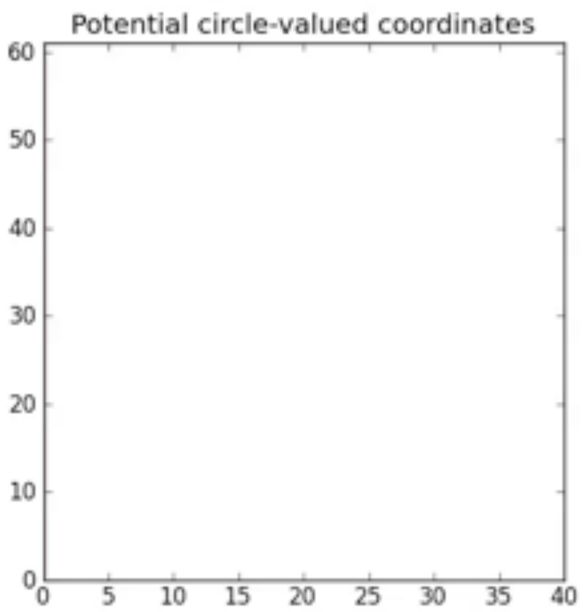
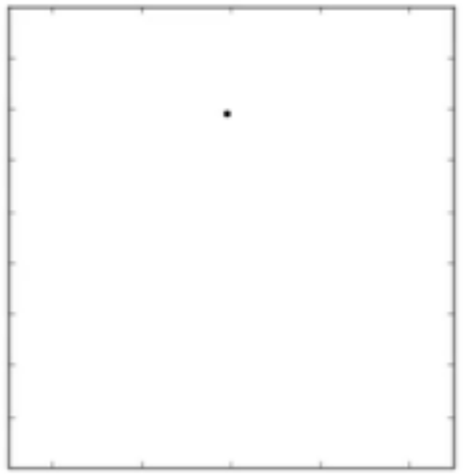
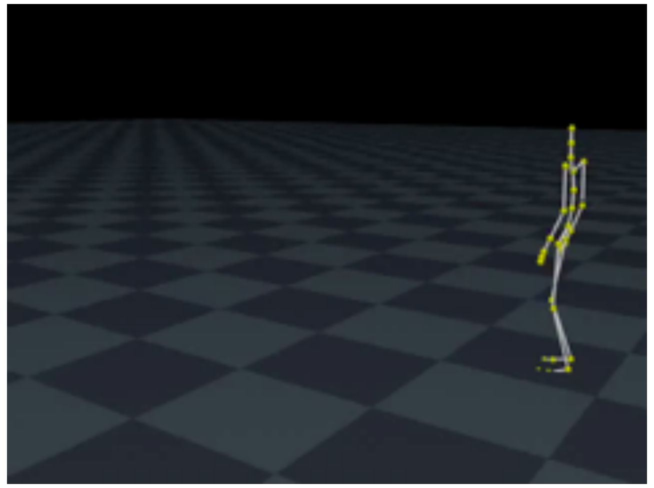
Cohomology

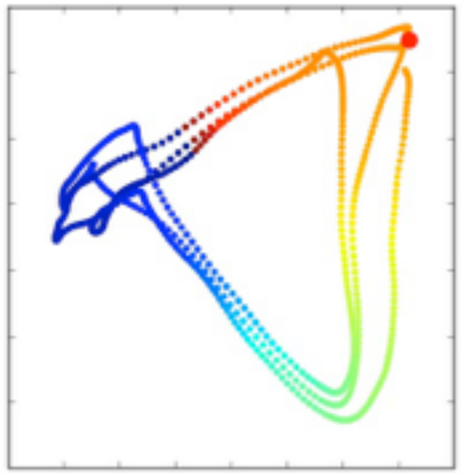
- Produces a topological system for generating circle-valued coordinate functions.
- This improves on state-of-the-art for analyzing recurrence.
- Applicable in motion capture and gait analysis.

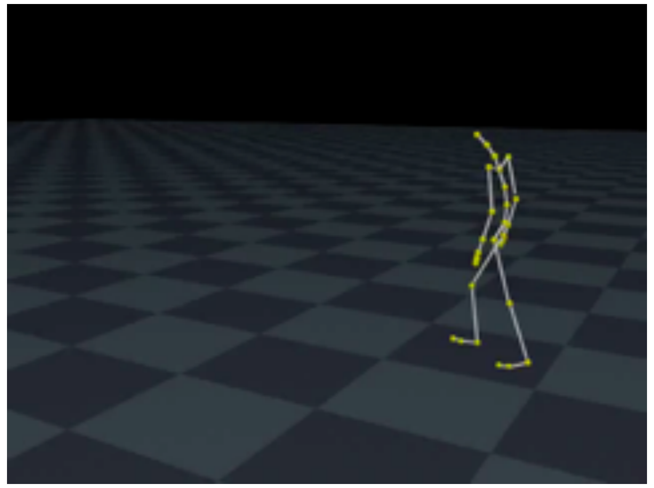












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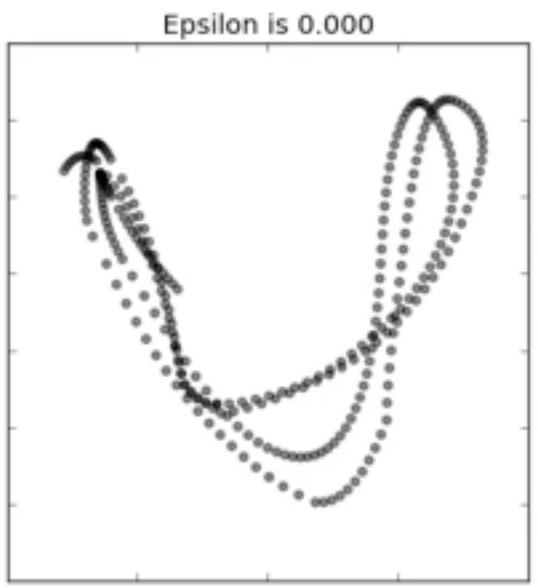

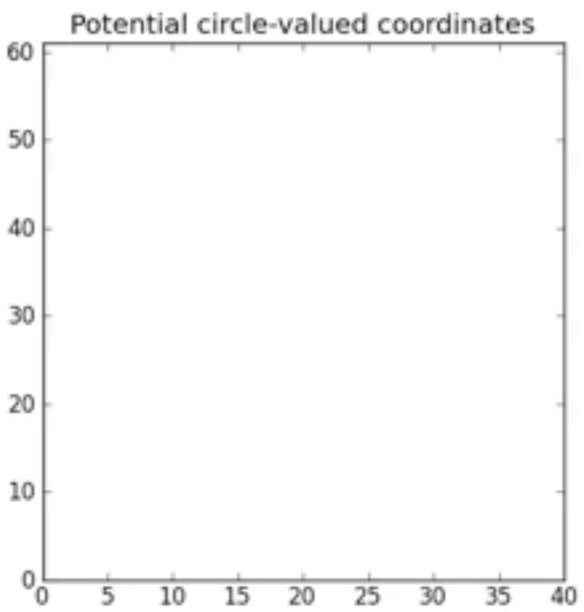

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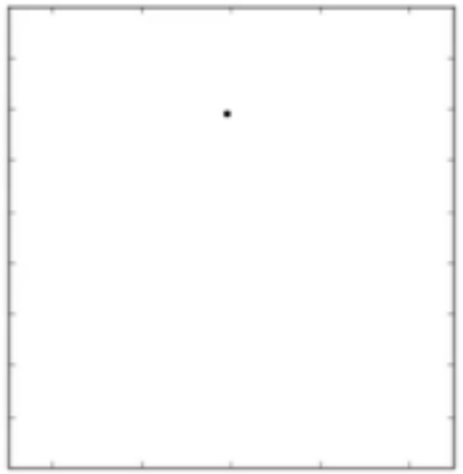

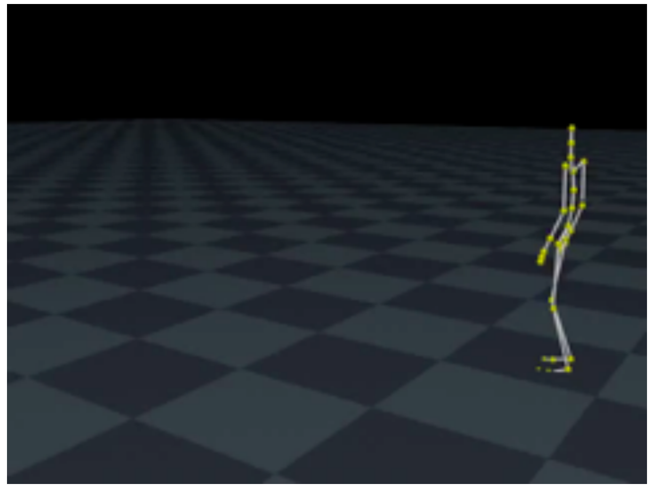

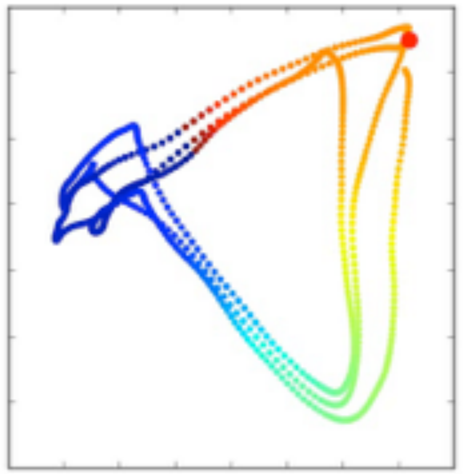

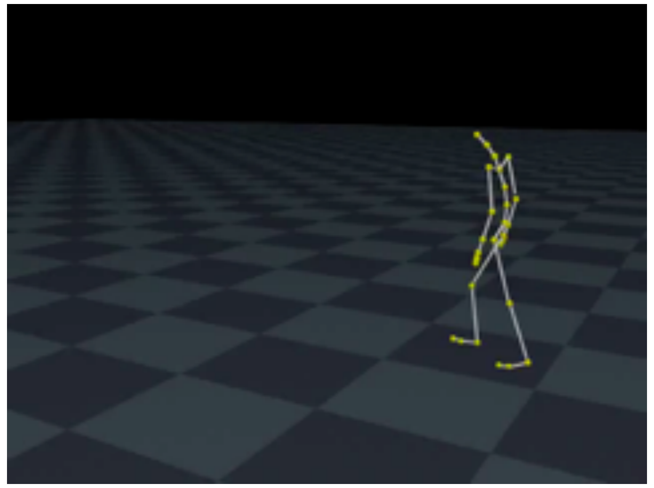

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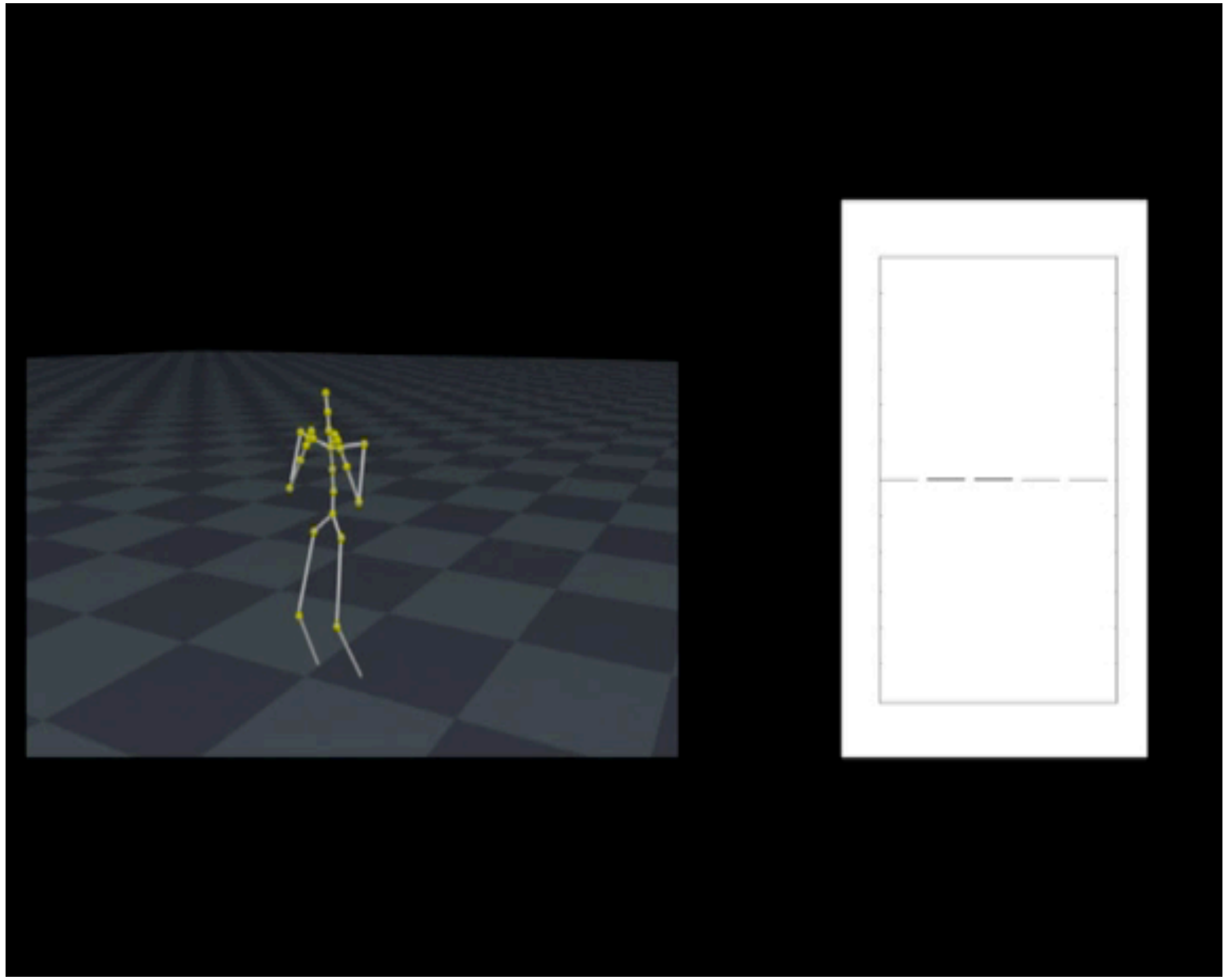
Indicator / feature functions on recurrence

- The functions generated by cohomology work as indicators of different recurrent features.



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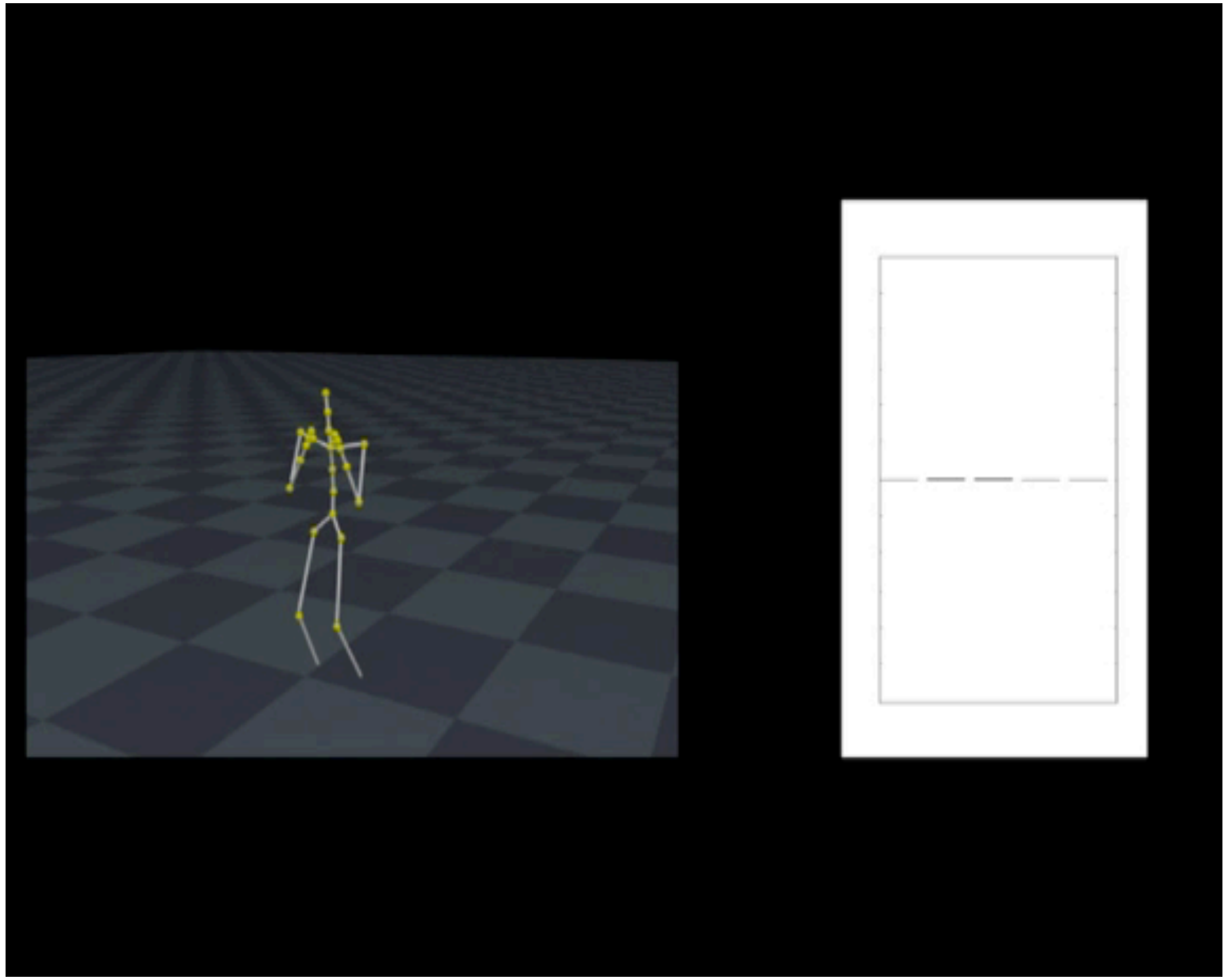
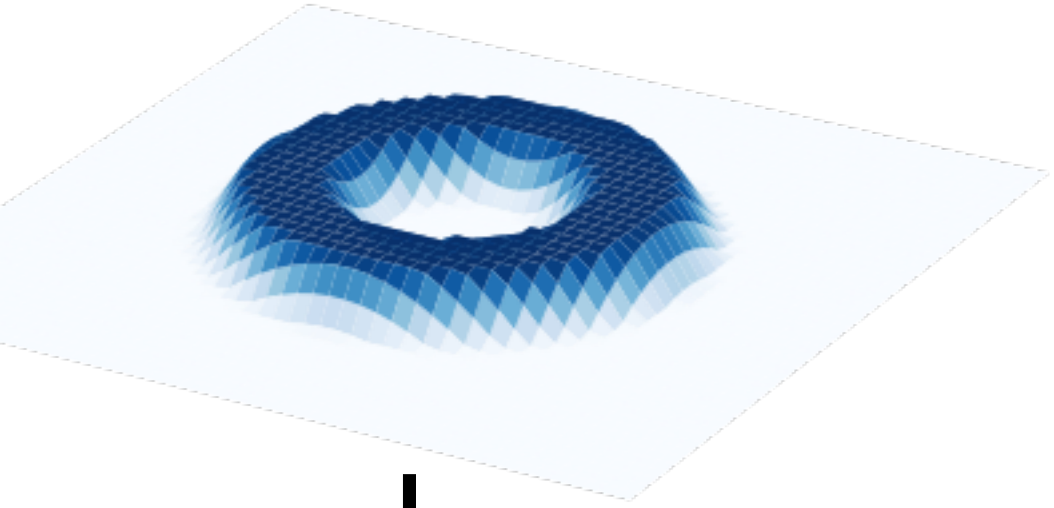


Idea for a pipeline for online topological learning



Online density estimation

$\varphi(a_t)$

$\varphi(a_t)$

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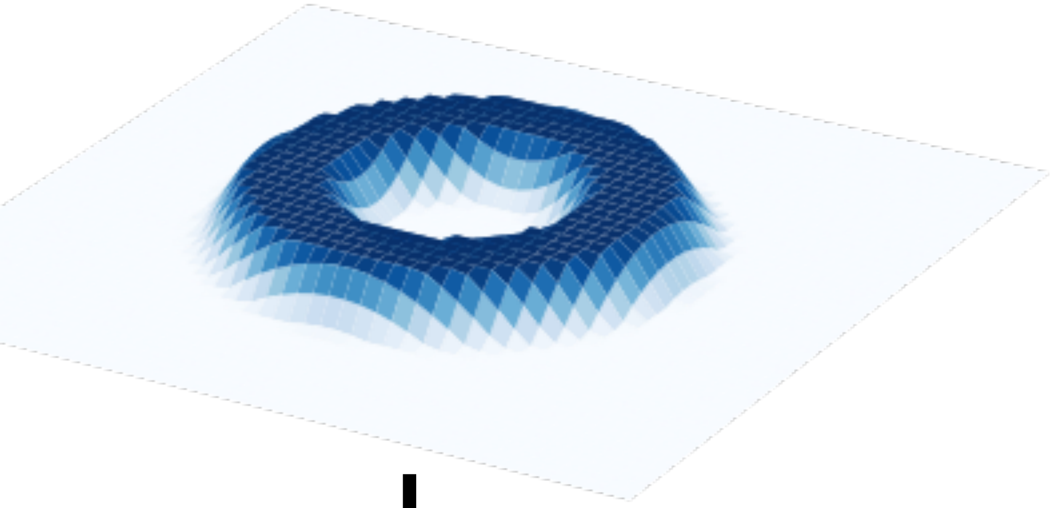

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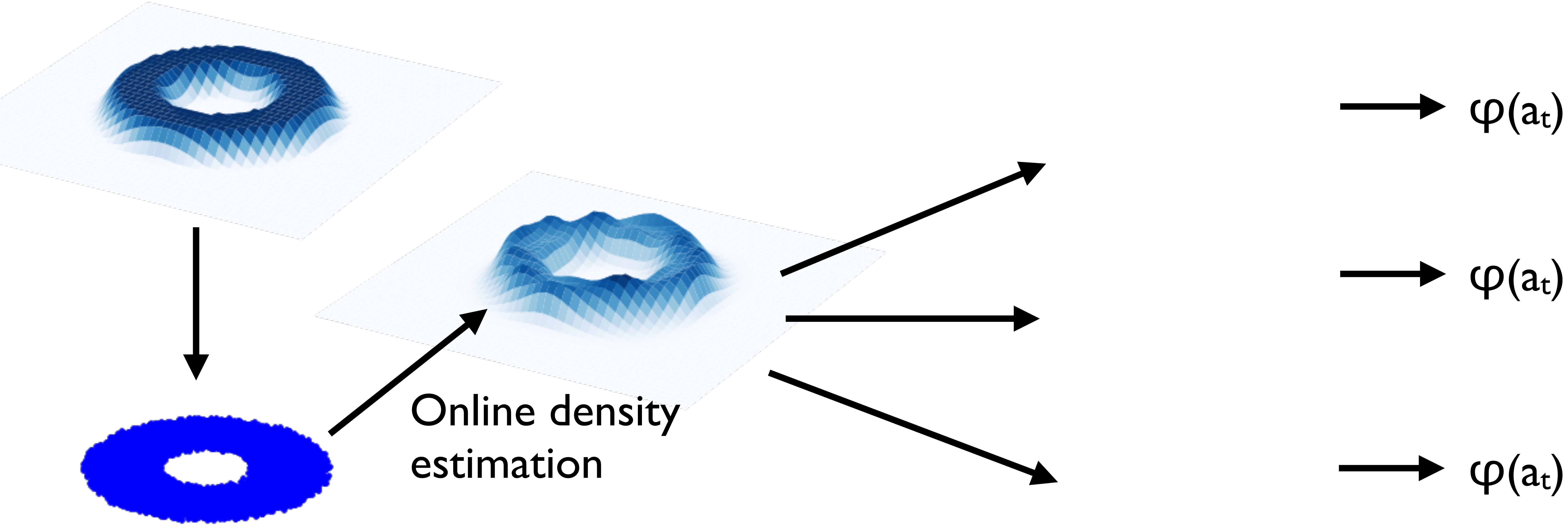

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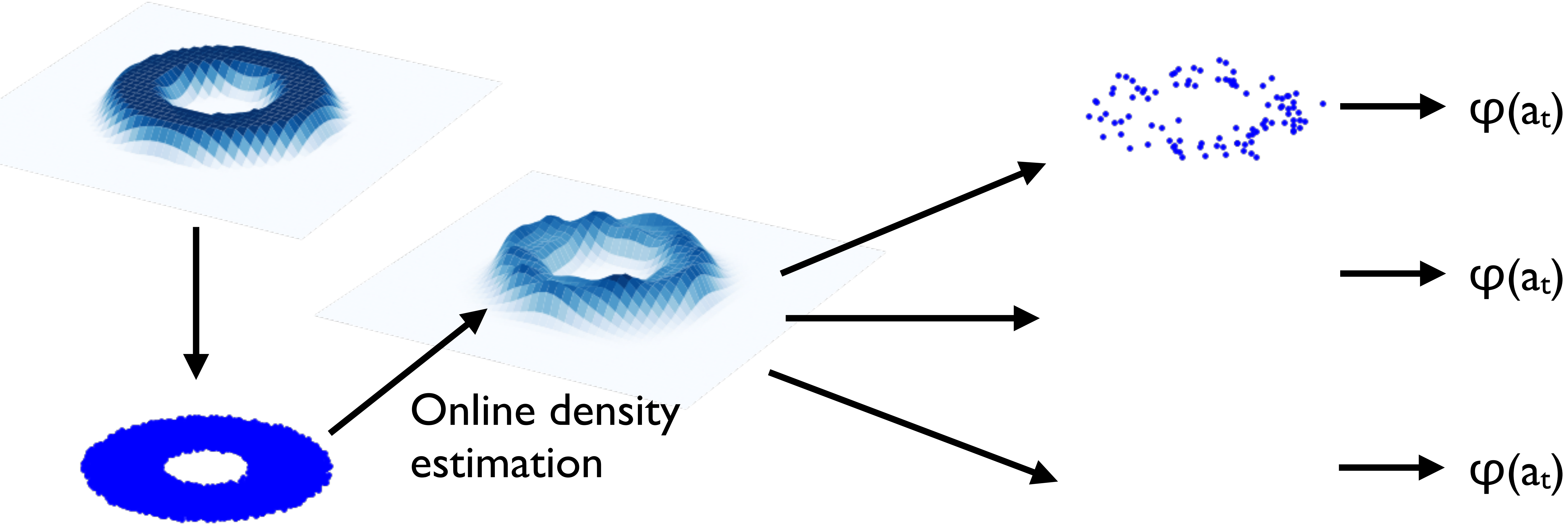

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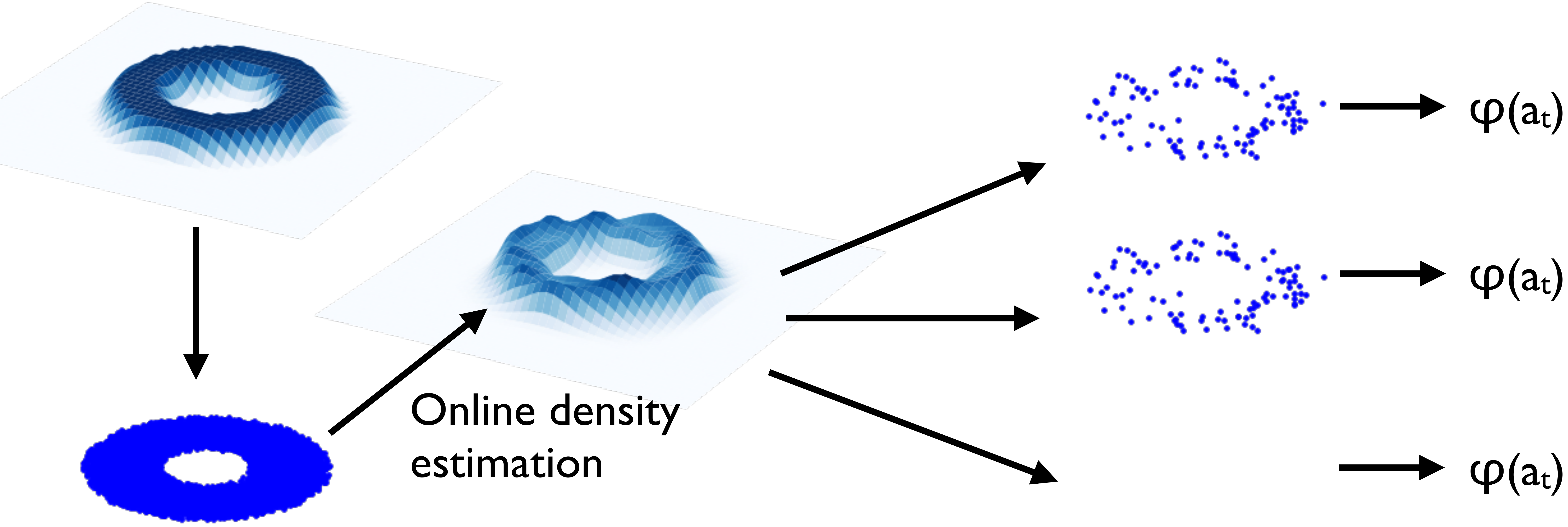

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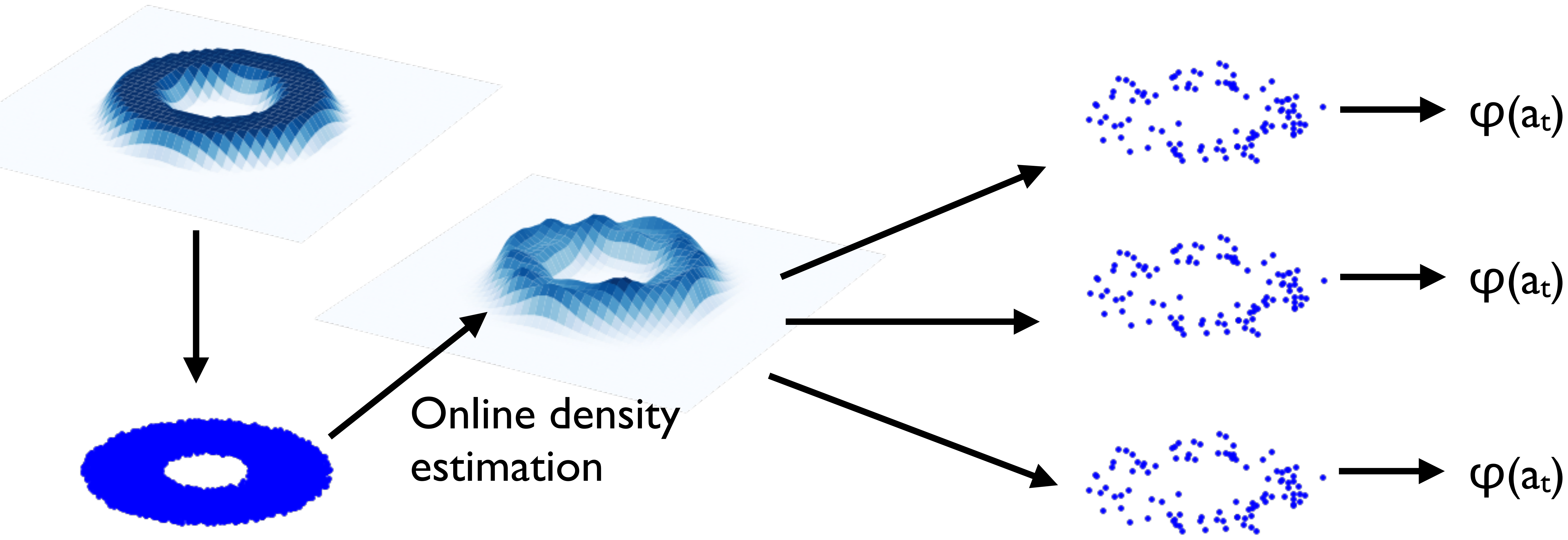
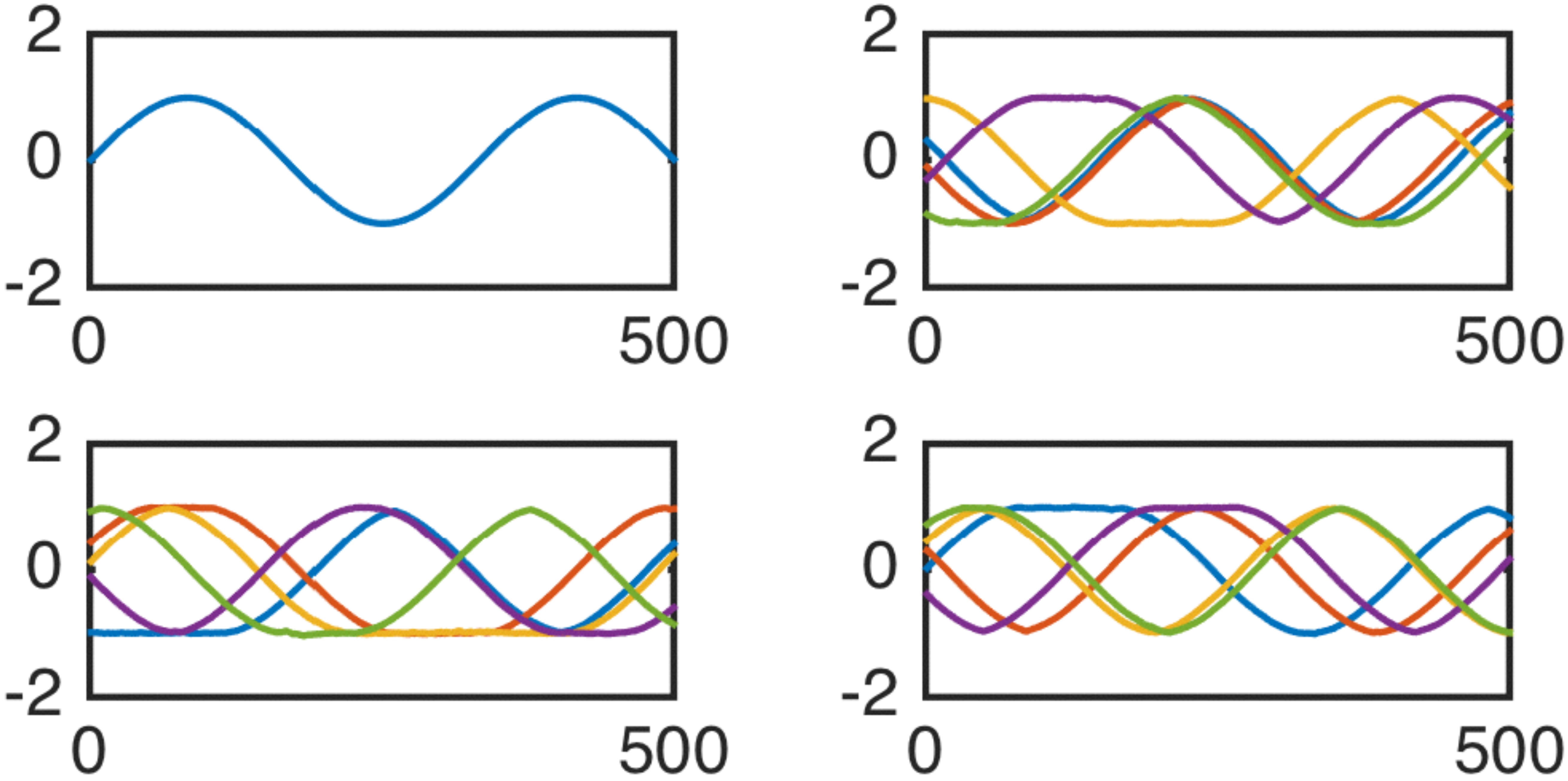


Current state



1-d seems doable: learned curves qualitatively similar

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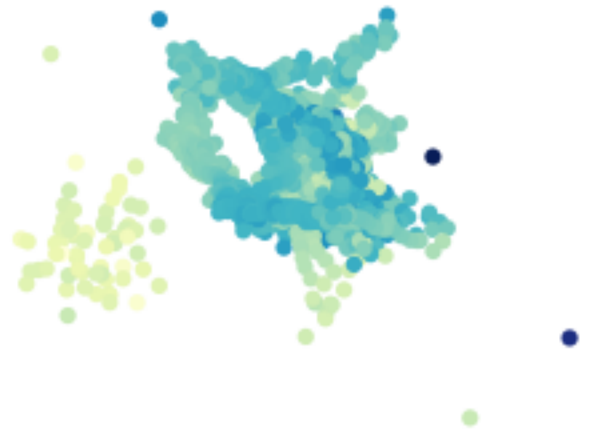
Current state

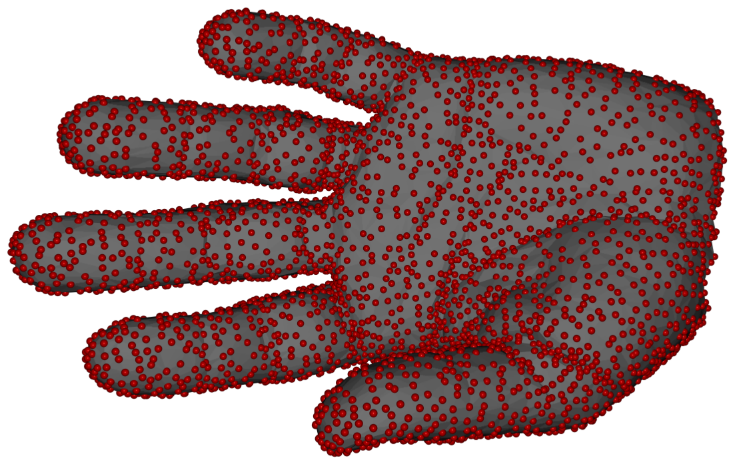


3d with Gaussian Mixture Models? Not quite…

Next up: try GPLVM
Gaussian Process Latent Variable Models

If anyone has any good ideas — **tell me**!

Mapper



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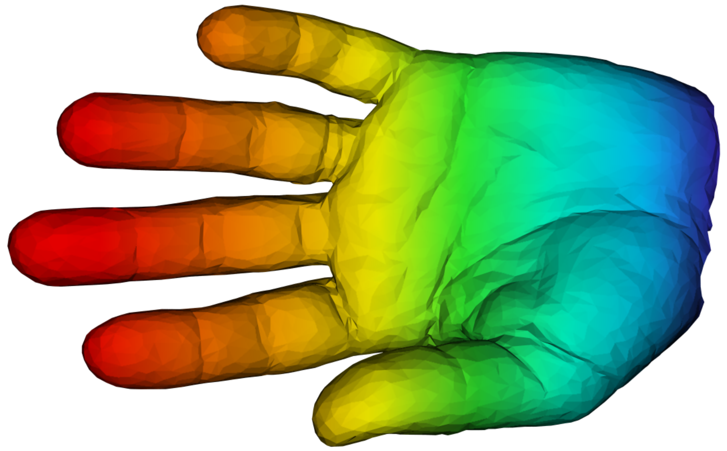
Mapper



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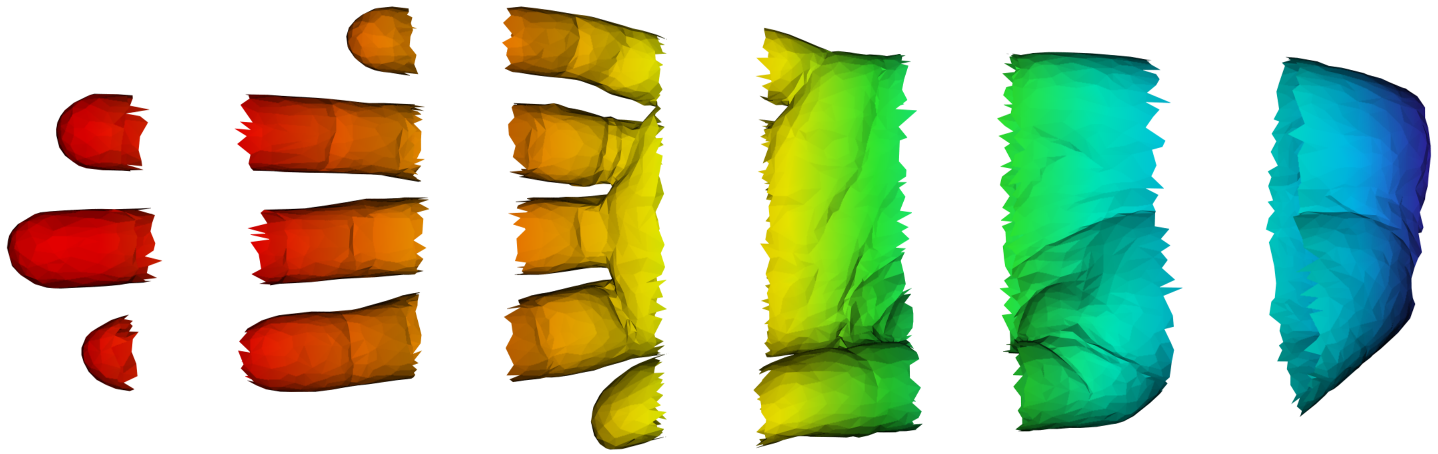
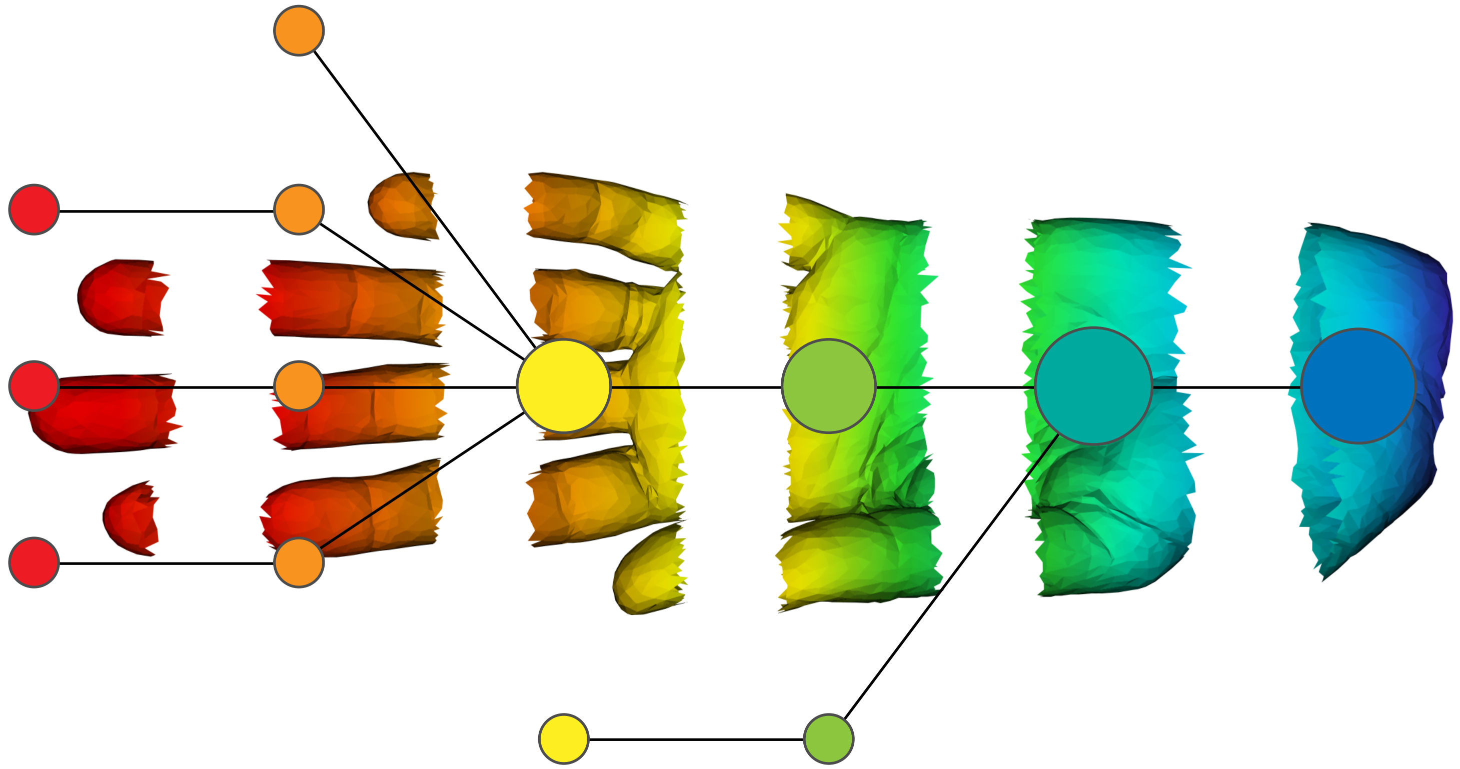
Mapper



Mapper



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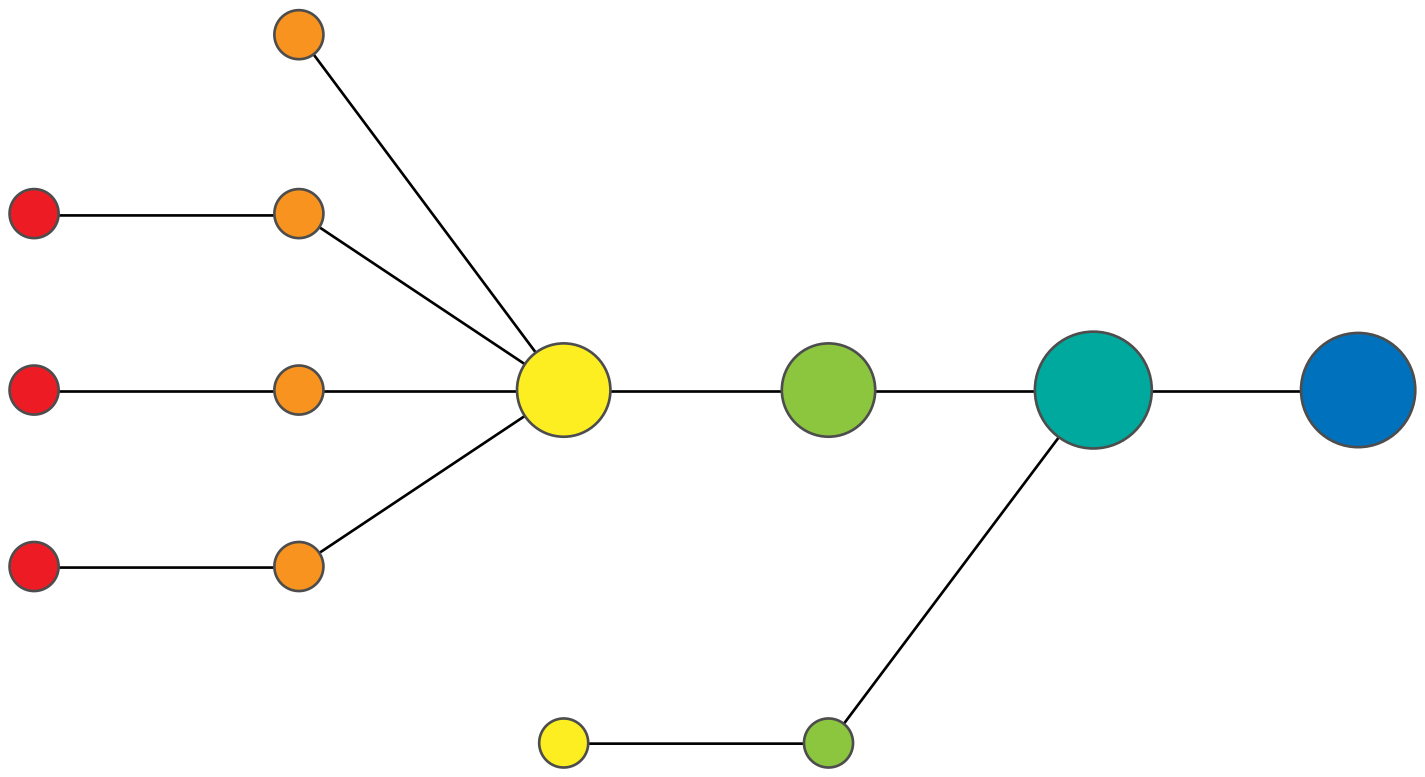
Mapper



Mapper

- Builds automatically a discrete model.
- Stable under permutation of data.
  Under permutation of filter functions.
- Persistent topology along varying feature functions:
  active research field.
- Strong potential for novel model creation for ML application.
  Going beyond Data Analysis.

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Thank you for listening

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